Big Heegner point Kolyvagin system for a family of modular forms

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ABSTRACT. The principal goal of this paper is to develop Kolyvagin's descent to apply with the big Heegner point Euler system constructed by Howard for the big Galois representation $\mathbb{T}$ attached to a Hida family $\mathbb{F}$ of elliptic modular forms. In order to achieve this, we interpolate and control the Tamagawa factors attached to each member of the family $\mathbb{F}$ at bad primes, which should be of independent interest. Using this, we then work out the Kolyvagin descent on the big Heegner point Euler system so as to obtain a big Kolyvagin system that interpolates the collection of Kolyvagin systems obtained by Fouquet for each member of the family individually. This construction has standard applications to Iwasawa theory, which we record at the end.

CONTENTS

2000 *Mathematics Subject Classification.* 11G05; 11G10; 11G40; 11R23; 14G10.
Key words and phrases. Heegner Points, Deformations of Kolyvagin systems, Iwasawa's main conjecture.

1. Introduction

The main goal of this article is to develop a Kolyvagin descent procedure for the big Heegner point Euler system constructed by Howard in [How07], associated to a Hida family of ordinary modular forms. This we achieve under the hypothesis that the family passes through a single (twisted) eigenform whose Tamagawa factors at bad primes are coprime to p. Through this construction, we obtain a big Kolyvagin system for the big Galois representation, with standard applications. Before stating our results, we start with setting up the notation.

Let N be a positive integer and $p \nmid 6N$ a prime. Define

$$\omega : \Delta = (\mathbb{Z}/p\mathbb{Z})^{\times} \longrightarrow \boldsymbol{\mu}_{p-1}$$

to be the Teichmüller character, which we view both as a p-adic and complex character by fixing embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, as well as a Dirichlet character modulo Np. Let

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(Np), \omega^j)$$

be a normalized cusp form of weight $k \geq 2$, which is an eigenform for the Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid Np$. Let $E/\mathbb{Q}_p$ be a finite extension that contains a_n for all n and let $\mathcal{O} = \mathcal{O}_E$ be its ring of integers and $\pi = \pi_E$ a fixed uniformizer. We assume further that f is an eigenform that is p-ordinary and p-stabilized, and the conductor of f is divisible by N. This amounts to saying that $a_p \in \mathcal{O}^{\times}$ and the system of Hecke-eigenvalues $\{a_\ell \mid \ell \nmid Np\}$ associated to f agrees with that of a newform of level N or Np. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be the Galois representation attached to f by Deligne [Del71]. Throughout this paper, we assume the following holds:

Hypothesis 1.1. The semi-simple residual representation $\overline{\rho}_f$ associated to ρ_f is absolutely irreducible and is p-distinguished.

Let $\Gamma = 1 + p\mathbb{Z}_p$. Identify Δ by $\boldsymbol{\mu}_{p-1}$ via ω so that we have

$$\mathbb{Z}_p^{\times} \cong \Delta \times \Gamma.$$

Set $\Lambda = \mathcal{O}[[\Gamma]]$. Let $\mathfrak{h}^{\mathrm{ord}}$ be Hida's universal ordinary algebra parametrizing the Hida family passing through f, which is finite flat over Λ by [Hid86a, Theorem 1.1]. We will recall some basic properties of $\mathfrak{h}^{\mathrm{ord}}$, for details the reader may consult [Hid86a, Hid86b] and [EPW06] for an excellent quick survey. The eigenform f fixed as above corresponds to an *arithmetic specialization* (in the sense of Definition 2.1 below)

$$\mathfrak{s}_f : \mathfrak{h}^{\mathrm{ord}} \longrightarrow \mathcal{O}$$

Decompose $\mathfrak{h}^{\mathrm{ord}}$ into a direct sum of its completions at maximal ideals and let $\mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ be the (unique) summand through which $\mathfrak{s}_f$ factors. The localization of $\mathfrak{h}^{\mathrm{ord}}$ at $\ker(\mathfrak{s}_f)$ is a discrete valuation ring [Nek06, §12.7.5], and hence there is a unique minimal prime $\mathfrak{a} \subset \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ such that $\mathfrak{s}_f$ factors through the integral domain

$$\mathcal{R} = \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}/\mathfrak{a}. \tag{1.1}$$

The Λ-algebra $\mathcal{R}$ is called the branch of the Hida family on which f lives, by duality it corresponds to a family $\mathbb{F}$ of ordinary modular forms. Hida [Hid86b] gives a construction of a big $G_{\mathbb{Q}}$-representation $\mathbf{T}$ with coefficients in $\mathcal{R}$, the exact definition of $\mathbf{T}$ is recalled below. Thanks to Hypothesis 1.1, $\mathbf{T}$ is a free $\mathcal{R}$-module of rank two; note that the assumption that $\overline{\rho}_f$ is p-distinguished is erroneously missing in [MT90]. The $G_{\mathbb{Q}}$-representation $\mathbf{T}$ is unramified outside Np. Let $\mathbb{T}$ be the critical twist of $\mathbf{T}$, as defined by Howard [How07, Definition 2.1.3]. Then the $G_{\mathbb{Q}}$-representation is self-dual in the sense that there is a perfect $\mathcal{R}$-bilinear pairing

$$\mathbb{T} \times \mathbb{T} \longrightarrow \mathcal{R}(1).$$

Fix a quadratic imaginary number field K and let $\mathcal{O}_K$ b its ring of integers. Assume until the end that the following holds:

Hypothesis 1.2.

(i) There is an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$.
(ii) The class number of K is prime to p.

Let $H_{\mathfrak{c}}$ be the ring class field of K of conductor $\mathfrak{c}$ and for c prime to p (resp., for $\alpha \in \mathbb{Z}^+$), let $K(c)$ (resp., K_α) be the maximal p-extension in H_c/K (resp., in $H_{p^{\alpha+1}}/K$). Set $K_\alpha(c)$ to be the composite field of K_α and $K(c)$, $K_\infty = \cup_\alpha K_\alpha$, $\Gamma^{\mathrm{ac}} = \mathrm{Gal}(K_\infty/K)$ and $\Lambda^{\mathrm{ac}} = \mathbb{Z}_p[[\Gamma^{\mathrm{ac}}]]$. In [How07, §2.2], Howard constructs a family of cohomology classes $\mathfrak{X}_{\mathfrak{c}} \in \tilde{H}^1_f(H_{\mathfrak{c}}, \mathbb{T})$, where $\tilde{H}^1_f(H_{\mathfrak{c}}, \mathbb{T})$ is Nekovář's [Nek06, §6] extended Selmer group. Howard also checks in Proposition 2.3.1 of loc.cit. that these classes satisfy the Euler system relation. For c prime to p, set

$$\mathfrak{z}_{c,\alpha} = \mathrm{cor}_{H_{cp^{\alpha+1}}/K_\alpha(c)}\left(U_p^{-\alpha}\mathfrak{X}_{cp^{\alpha+1}}\right) \in \tilde{H}^1_f(K_\alpha(c), \mathbb{T}).$$

This definition makes sense thanks to [How07, Proposition 2.3.1]. The collection $\{\mathfrak{z}_{c,\alpha}\}_{c,\alpha}$ is called the *big Heegner point Euler system.* To ease notation, write

$$\mathfrak{z}_\alpha = \mathfrak{z}_{1,\alpha} \in \tilde{H}^1_f(K_\alpha, \mathbb{T}).$$

The collection $\{\mathfrak{z}_\alpha\}$ is norm-compatible as α varies and we may therefore set

$$\mathfrak{z}_\infty = \{\mathfrak{z}_\alpha\} \in \varprojlim_\alpha \tilde{H}^1_f(K_\alpha, \mathbb{T}) =: \tilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}).$$

The first (**H.stz**) of the following hypotheses may be thought of as an assumption to rule out the existence of exceptional zeros (in the sense of Greenberg [Gre94]) at characters of Γ^{ac} of finite order. The second (**H.Tam**) has to do with Tamagawa factors.

(**H.stz**) For every $v|p$, $H^0(K_v, \mathrm{F}_v^-(\overline{T})) = 0$.

Here $\overline{T} = \mathbb{T}/\mathfrak{m}_{\mathcal{R}}$ is the residual representation and $\mathrm{F}_v^-(\overline{T})$ is defined as in §4. See Remark 4.24 for the content of this hypothesis.

(**H.Tam**)

(i) $p \nmid \prod_{\ell|N}(\ell^2 - 1)$,

(ii) there is a specialization T of the twisted Hida family $\mathbb{T}$ for which $p \nmid c_\ell(T)$.

Starting from an Euler system for a Galois representation M with coefficients in a discrete valuation ring, Mazur and Rubin [MR04] devise[1] a machinery which yields a *Kolyvagin system* for M. This is what we carry out for the big Galois representation $\mathbb{T}$ which has coefficients over a dimension-2 Gorenstein ring $\mathcal{R}$ and prove the following:

Theorem A.1.[See Theorem 4.28] *Suppose the assumptions* **H.Tam** *and* **H.stz** *hold true. There is a Kolyvagin system*

$$\{\kappa_n\} = \boldsymbol{\kappa} \in \overline{\boldsymbol{KS}}(\mathbb{T} \otimes \Lambda^{\mathrm{ac}}, \mathcal{F}_{\mathrm{Gr}})$$

(where the $\mathcal{R}$-module $\overline{\boldsymbol{KS}}(\mathbb{T} \otimes \Lambda^{\mathrm{ac}}, \mathcal{F}_{\mathrm{Gr}})$ is described in Definition 4.12 below) such that

$$\kappa_1 = \mathfrak{z}_\infty \in \tilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}).$$

Assuming (**H.Tam**)(ii) alone, Howard in [How07, Theorem 2.4.5] proves that the classes $\{\mathfrak{z}_{c,\alpha}\}$ lie in the Greenberg Selmer group. However, to carry out the descent argument (as we do in §4) in order to deduce Theorem A.1, one needs the finer analysis of local cohomology groups that we carry out in §3. As a by-product to our analysis we control, among other things, the

[1]Attentive reader will notice that Mazur and Rubin never treat Heegner points. It was Howard [How04] who was the first to study the Heegner points from the perspective offered by the work of Mazur and Rubin.

variation of Tamagawa factors in the "Hida family" $\mathbb{T}$, much in the spirit of [EPW06]. To that end, we show for a prime ℓ that divides the tame conductor N, how to interpolate the Tamagawa factors $\{c_\ell(T_S)\}_{\mathfrak{s}}$ into an element $\boldsymbol{\tau}$ of $\mathcal{R}$ (which we call the *Tamagawa element*). Here $\mathfrak{s}$ runs through specializations

$$\mathfrak{s} : \mathcal{R} \longrightarrow S, \qquad T_S = \mathbb{T} \otimes_{\mathcal{R}} S$$

into discrete valuation rings S and the Tamagawa factor $c_\ell(T_S)$ is defined following Fontaine and Perrin-Riou [FPR94]. More precisely, we prove the following in §3:

Theorem A.2.

(i) *There exists an element $\boldsymbol{\tau} \in \mathcal{R}$ such that $[S : \mathfrak{s}(\boldsymbol{\tau})] = c_\ell(T_S)$.*

(ii) *Assume that the hypothesis* **H.Tam** *holds true. Then for any specialization T_S of $\mathbb{T}$, the Tamagawa factor $c_\ell(T_S)$ is coprime to p as well.*

See §3 below for a precise definition of the element $\boldsymbol{\tau}$ of $\mathcal{R}$, which might be of independent interest. We remark that Theorem A.2(ii) can be obtained without going through the construction of the element $\boldsymbol{\tau}$, c.f., [EPW06, Propositions 2.2.4 and 2.2.5], [Och06, Theorem 3.3], [FO12, Lemma 2.14]. However, in order to descend to a Kolyvagin system, one needs the finer analysis in §3.

Once we obtain a Kolyvagin system as in Theorem A.1, a standard argument (c.f., [Och05, Fou10]) gives bounds on the appropriate extended Selmer group. Suppose the ring $R_\infty := \mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda^{\mathrm{ac}}$ is a regular ring and M is a torsion R_∞-module. We define the characteristic ideal of M to be

$$\mathrm{char}(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{length}(M_{\mathfrak{p}})}$$

where the product runs through height-1 primes of R_∞.

Theorem A.3.[See Theorem 5.1] *Suppose the assumptions of Theorem A.1 hold true and assume that the ring R_∞ is regular. Then*

$$\mathrm{char}\left(\tilde{H}^2_{f,\mathrm{Iw}}(K_\infty, \mathbb{T})_{\mathrm{tors}}\right) \mid \mathrm{char}\left(\tilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T})/R_\infty \mathfrak{z}_\infty\right)^2.$$

We note that Fouquet has also devised a machinery to make use of the big Heegner point Euler system and has obtained Theorem A.3 [Fou10, Theorem 2.9]. In his statement, however, the class $\mathfrak{z}_\infty$ here has to be replaced by a multiple by an element $\alpha \in \mathcal{R}$, accounting for uncontrolled Tamagawa factors. We are able to control the uncontrolled "extra" factor α thanks to Theorem A.1 and thus improve on Fouquet's result towards Howard's two-variable main conjecture. In fact, we expect that the big Kolyvagin system

$\boldsymbol{\kappa}$ we construct in Theorem A.1 (under the hypotheses **H.stz** and **H.Tam**) is *primitive* in a certain sense and that the divisibility in Theorem A.1 is sharp, which is not the case in [Fou10]. All this is a consequence of the fact that Fouquet's approach is all together different than ours: Fouquet constructs Kolyvagin systems for each individual specialization, whereas in this paper we prove the existence of a big Kolyvagin system that essentially interpolates each of these individual Kolyvagin systems.

We finally remark that the hypothesis **H.stz** seems to be also necessary for the descent arguments in [Fou10] to go through, see Remark 4.24 below.

When the base field is a general totally real field, there also exists a Hida family of Hilbert modular forms and the relevant properties of the associated big Galois representation is established in [Hid88, Hid89, SW99, SW01]. A construction of an Euler system of big Heegner points in this setting is due to Fouquet [Fou08], generalizing the work of Howard. The formalism of the current article applies to this more general setting as well, and that is one reason why the author chose to stick to the case of elliptic modular forms so as to keep the notation simple and various technical constructions tractable.

Our construction of a big Heegner point Kolyvagin system (Theorem A.1) goes hand in hand with the main result of the forthcoming article [Büy12], where one may better observe the benefits of deforming Kolyvagin systems directly (as opposed to deforming first the Euler system and then specializing to Kolyvagin systems for each individual member, as done so in [Fou10].) In [Büy12], we prove (generalizing the main theorem of [Büy11]) that the $\mathcal{R}$-module of Kolyvagin systems for $\mathcal{T}$ is free of rank one for a very general class of Galois representations $\mathcal{T}$ with coefficients

- either a 2-dimensional complete Gorenstein ring $\mathcal{R}$,
- or a regular complete Noetherian local ring $\mathcal{R}$.

In particular, this result may be used to interpolate Kolyvagin systems obtained from Kato's Euler systems for elliptic modular forms to the universal deformation ring, under mild hypotheses. Such a construction seems intractable for the time being in the level of Euler systems. Furthermore, one may hope to extend the arguments of [Büy12] in order to interpolate Kato's Kolyvagin systems for each individual member of a finite slope (not necessarily p-ordinary) Coleman family, and incorporate this construction within Pottharst's [Pot11] non-ordinary Iwasawa theory.

1.1. **Notation and Hypotheses.** For any field F, fix a separable closure $\overline{F}$ of F and write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. For a continuous G_F-representation M, we will denote by $H^i(F,M) = H^i(G_F,M)$ the cohomology group calculated with continuous cochains.

For an algebraic number field L and a non-archimedean place w of L, we write G_w instead of G_{L_w}. We also denote any fixed decomposition subgroup (resp., inertia subgroup) at w of G_L by D_w (resp., I_w).

For an abelian group A, let $\hat{A} = \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$ be its p-adic completion. If further G_F acts on A, then for a character χ of G_F with values in $\mathcal{O}$ (where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$), let

$$A^{\chi} := \{a \in \hat{A} \otimes_{\mathbb{Z}_p} \mathcal{O} : g \cdot a = \chi(g)a \text{ for all } g \in G_F\}.$$

2. A family of Galois representations

In this section, we give the definition of the big Galois representation $\mathbb{T}$ attached to a Hida family of elliptic modular forms $\mathbb{F}$. We mostly follow [How07, §2] as we will heavily rely on his constructions later in this paper, and we use in this section the terminology set in loc. cit. sometimes without giving the definition here.

Let $\mathfrak{h}^{\mathrm{ord}}$ be Hida's big ordinary Hecke algebra of tame level N and let $\Lambda = \mathcal{O}[[\Gamma]]$ be the Iwasawa algebra as in the introduction. Then $\mathfrak{h}_{\mathrm{ord}}$ can made into a Λ-algebra via the diamond action and it is finite flat over Λ by [Hid86a, Theorem 1.1].

Definition 2.1.

(i) Let ι be the natural inclusion $\mathbb{Z}_p^{\times} \to \mathcal{O}[[\mathbb{Z}_p^{\times}]]^{\times}$. The restriction of ι to Γ will also be denoted by ι.

(ii) If A is any finitely generated commutative Λ-algebra then the $\mathcal{O}_E$-algebra map $A \xrightarrow{s} \overline{\mathbb{Q}}_p$ is called an *arithmetic specialization* if the composition

$$\Gamma \xrightarrow{\iota} A^{\times} \xrightarrow{s} \overline{\mathbb{Q}}_p^{\times}$$

has the form $\gamma \mapsto \psi(\gamma)\gamma^{r-2}$ for some integer $r \geq 2$ and some character ψ of Γ of finite order.

(iii) The kernel of an arithmetic specialization is called an *arithmetic prime* of A. If $\wp$ is an arithmetic prime then the residue field $E_{\wp} := A_{\wp}/\wp A_{\wp}$ is a finite extension of E. The composition

$$\Gamma \longrightarrow A^{\times} \longrightarrow E_{\wp}^{\times}$$

has the form $\gamma \mapsto \psi_{\wp}(\gamma)\gamma^{r-2}$ for a character $\psi_{\wp} : \Gamma \to E_{\wp}^{\times}$. The character $\psi_{\wp}$ is called the wild character of $\wp$ and the integer r is called the *weight* of $\wp$.

Let $f \in S_k(\Gamma_0(Np), \omega^j)$ be a cusp form as in the introduction. As explained in Introduction, f corresponds to an arithmetic specialization $\mathfrak{s}_f$

which factors through

$$\mathcal{R} = \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}/\mathfrak{a}$$

for a unique maximal ideal $\mathfrak{m} \subset \mathfrak{h}^{\mathrm{ord}}$ and a uniquely determined minimal prime ideal $\mathfrak{a} \subset \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$.

For $s \in \mathbb{Z}^+$, set $\Phi_s = \Gamma_0(N) \cup \Gamma_1(p^s) \subset \mathrm{SL}_2(\mathbb{Z})$ and let Y_s denote the affine modular curve Y_s classifying elliptic curves with Φ_s-level structure. More precisely, Y_s classifies triples (E, C, π) consisting of an elliptic curve E, a cyclic subgroup C of E of order N and a point π on E of exact order p^s. Let X_s be its compactification and let J_s be the Jacobian of X_s. Then there is a degeneracy map

$$\alpha : X_{s+1} \longrightarrow X_s$$

given by

$$(E, C, \pi) \mapsto (E, C, p \cdot \pi),$$

which induces a map

$$\alpha_* : J_{s+1} \longrightarrow J_s.$$

Let $T_p(J_s)$ be the p-adic Tate module. Via the Albenese action, Hecke operators acts on each $T_p(J_s)$. Let $e^{\mathrm{ord}} = \lim U_p^{n!}$ be Hida's ordinary projector, set $T_p^{\mathrm{ord}}(J_s) = e^{\mathrm{ord}}(T_p(J_s))$ and

$$\mathbf{T} = (\varprojlim T_p^{\mathrm{ord}}(J_s)) \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathcal{R}.$$

Here the inverse limit is with respect to α_*. There is a natural $\mathcal{R}$-linear $G_{\mathbb{Q}}$-action on $\mathbf{T}$. As indicated in Introduction, $\mathbf{T}$ is a free $\mathcal{R}$-module of rank two and is unramified outside Np. Let $\mathbb{T}$ be the self-dual twist of $\mathbf{T}$, defined as in [How07, Definition 2.1.3] (and denoted by $\mathbf{T}^\dagger$ in loc. cit.).

For any arithmetic prime $\wp \subset \mathcal{R}$, Hida Theory associates an ordinary modular form $f_\wp$ with coefficients in $E_\wp$. The $G_{\mathbb{Q}}$-representation $V_\wp = \mathbb{T} \otimes_{\mathcal{R}} E_\wp$ is then a self-dual twist of the p-adic Galois representation attached to the form $f_\wp$ by Deligne.

3. Controlling the Tamagawa factors attached to families

Let $\mathcal{R}$ and $\mathbb{T}$ be as above and let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{R}$. We write $\mathcal{L} = \mathrm{Frac}(\mathcal{R})$, the field of fractions of $\mathcal{R}$ and $\mathbb{V} = \mathbb{T} \otimes_{\mathcal{R}} \mathcal{L}$. Throughout this section $v \nmid p\infty$ will denote a place of K which divides the tame conductor N of the Hida family $\mathcal{R}$.

For technical reasons we also impose the following:

Assumption 3.1. $p \nmid \prod_{v|N} (\mathbf{N}v^2 - 1).$

3.1. **Local Galois representation at the primes dividing the tame conductor.** Nekovář in [Nek06, Proposition 12.7.14.1] describes the $\mathcal{L}[G_v]$-module $\mathbb{V}$ under the following assumption:

($\mathrm{H_v}$) There is a twisted cusp form $g_\wp$ through which the (twisted) Hida family passes through (in the sense of [Nek06, §12.7.10]) such that $\pi(g_\wp)_v = \mathrm{St}(\mu)$ (with $\mu^2 = 1$ and unramified).

Here $\pi(g_\wp)_v$ is the smooth admissible representation of $\mathrm{GL}_2(\mathbb{Q}_v)$ attached to $g_\wp$ at v and $\mathrm{St}(\mu)$ is the twisted Steinberg, $\mu : K_v^\times \to \mathbb{C}^\times$ is a character. See [Nek06, §12.3 and §12.7] for the content of the assumption ($\mathrm{H_v}$). Until the end, we assume that ($\mathrm{H_v}$) holds true.

Since we also assumed Hypothesis 1.1, [MT90, Théorème 7] shows that

$$\mathfrak{h}^{\mathrm{ord}} \cong \mathrm{Hom}_\Lambda(\mathfrak{h}^{\mathrm{ord}}, \Lambda).$$

It follows then, as explained in [Nek06, Proposition 12.7.14.1] that there is an exact sequence of $\mathcal{R}[[G_v]]$-modules

$$0 \longrightarrow F^+(\mathbb{T}) \longrightarrow \mathbb{T} \longrightarrow F^-(\mathbb{T}) \longrightarrow 0, \tag{3.1}$$

with

$$F^+(\mathbb{T}) \cong \mathcal{R}(1) \otimes \mu \quad \text{and,} \quad F^-(\mathbb{T}) \cong \mathcal{R} \tag{3.2}$$

as $\mathcal{R}[[G_v]]$-modules, where μ is an unramified character.

3.2. **The Tamagawa number.** Let Φ be a finite extension of $\mathbb{Q}_p$, and let $\mathfrak{O}$ denote its ring of integers. Suppose T is a $\mathfrak{O}[G_v]$-module such that there is an exact sequence of $\mathfrak{O}[G_v]$-modules

$$0 \longrightarrow \mathfrak{O}(1) \otimes \mu \longrightarrow T \longrightarrow \mathfrak{O} \longrightarrow 0. \tag{3.3}$$

Here, $\mu : K_v^\times \to \{\pm 1\}$ (which may be also thought as a character of the Galois group G_v via local class field theory). In what follows, we will compute the p-part of the Tamagawa number (which we denote by c_v) in terms of the sequence (3.3).

3.2.1. $\mu \neq \mathrm{id}$. Let L_w be the extension of K_v cut by μ, so that L_w/K_v is a quadratic extension. Let G_w denote the absolute Galois group of L_w and $\Delta = \mathrm{Gal}(L_w/K_v)$.

Proposition 3.2. *The following holds:*

(i) $H^1(K_v, \mathfrak{O}(1) \otimes \mu) = 0$,
(ii) $H^0(I_v, T \otimes \Phi/\mathfrak{O})$ *is divisible.*
(iii) $c_v = 1$.

Proof. Using the inflation-restriction sequence and Kummer theory, it follows that

$$H^1(K_v, \mathfrak{O}(1) \otimes \mu) = (L_w^\times)^{\mu^{-1}} \otimes_{\mathbb{Z}_p} \mathfrak{O}.$$

Here $(L_w^\times)^{\mu^{-1}}$ is the μ^{-1}-part of the $\mathbb{Z}_p[\Delta]$-module $L_w^{\times,\wedge}$, where $L_w^{\times,\wedge}$ is the p-adic completion of $L_w^\times$. Since $w \nmid p$ and since we assumed 3.1, we have a $\mathbb{Z}_p[\Delta]$-equivariant isomorphism

$$\mathrm{ord}_w : L_w^{\times,\wedge} \xrightarrow{\sim} \mathbb{Z}_p$$

with the trivial action on $\mathbb{Z}_p$. Since μ is a non-trivial character of Δ, it follows that $(L_w^\times)^{\mu^{-1}} = 0$ and thus (i) follows.

Now taking the G_v-cohomology of the sequence (3.3), using (i) and noting that the G_v-action on $\mathfrak{O} \otimes \mu$ is non-trivial, it follows that

$$T^{G_v} \xrightarrow{\sim} \mathfrak{O}.$$

This in return implies that the sequence (3.3) splits, yielding a decomposition

$$T = (\mathfrak{O}(1) \otimes \mu) \bigoplus \mathfrak{O}$$

as $\mathfrak{O}[G_v]$-modules. It now follows that $(T \otimes \Phi/\mathfrak{O})^{I_v} \cong \Phi/\mathfrak{O}$ (resp., $(\Phi/\mathfrak{O})^2$) if μ is ramified (resp., unramified).

Since $c_v = \#\left(H^0(I_v, T \otimes \Phi/\mathfrak{O})/H^0(I_v, T \otimes \Phi/\mathfrak{O})_{\mathrm{div}}\right)^{\mathrm{Fr}_v=1}$, (iii) follows from (ii). $\square$

3.2.2. $\mu = \mathrm{id}$. In this case the sequence (3.3) is

$$0 \longrightarrow \mathfrak{O}(1) \longrightarrow T \longrightarrow \mathfrak{O} \longrightarrow 0. \tag{3.4}$$

Let $\sigma = \partial(1) \in H^1(K_v, \mathfrak{O}(1))$ where $\partial : \mathfrak{O} \to H^1(K_v, \mathfrak{O}(1))$ is the connecting homomorphism in the long exact sequence of the G_v-cohomology of the sequence (3.4). Kummer theory gives an isomorphism

$$\mathrm{ord}_v : H^1(K_v, \mathbb{Z}_p(1)) \xrightarrow{\sim} K_v^{\times,\wedge} \xrightarrow{\sim} \mathbb{Z}_p,$$

which yields an isomorphism (which we still denote by ord_v) after tensoring by $\mathfrak{O}$

$$\mathrm{ord}_v : H^1(K_v, \mathfrak{O}(1)) \xrightarrow{\sim} \mathfrak{O}. \tag{3.5}$$

According to [CE56] pp. 290 and 292, $-\sigma$ is the extension class of the sequence (3.4) inside $\mathrm{Ext}^1_{\mathfrak{O}[G_v]}(\mathfrak{O}, \mathfrak{O}(1)) = H^1(K_v, \mathfrak{O}(1))$. Hence $\mathrm{ord}_v(\sigma) = 0$ if and only if the sequence (3.4) splits.

Proposition 3.3. *If the sequence (3.4) does not split, then*

$$c_v = \#(\mathfrak{O}/\mathrm{ord}_v(\sigma)\mathfrak{O})_{\mathrm{tors}}.$$

Proof. By functorialty, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathfrak{O}^{G_v} \arrow[r, equal] \arrow[d] & \mathfrak{O} \arrow[r, "\partial"] \arrow[d, equal] & H^1(K_v, \mathfrak{O}(1)) \arrow[d, "\cong"] & & \\
\mathfrak{O}^{I_v} \arrow[r, equal] & \mathfrak{O} \arrow[r, "\partial'"] & H^1(I_v, \mathfrak{O}(1)) \arrow[r, "\alpha"] & H^1(I_v, T) \arrow[r] & H^1(I_v, \mathfrak{O}) \cong \mathbb{Z}_p
\end{tikzcd}$$

The right most vertical arrow is an isomorphism because

- The cohomological dimension of G_v/I_v is one therefore the map
$$H^1(K_v, \mathfrak{O}(1)) \longrightarrow H^1(I_v, \mathfrak{O}(1))$$
is surjective,
- The group G_v/I_v is pro-cyclic and Fr_v is a topological generator, hence for the kernel $H^1(G_v/I_v, \mathfrak{O}(1))$ of this map we have:
$$H^1(G_v/I_v, \mathfrak{O}(1)) \cong \mathfrak{O}(1)/(\mathrm{Fr}_v - 1)\mathfrak{O}(1) \cong \mathfrak{O}/(\mathbf{N}v - 1)\mathfrak{O},$$
and since we assumed (3.1) this is trivial.

The second row of the diagram above shows that $H^1(I_v, T)_{\mathrm{tors}} = \mathrm{im}(\alpha)_{\mathrm{tors}}$. Furthermore, since G_v acts trivially on $H^1(I_v, \mathfrak{O}(1))$, it follows that $\mathrm{im}(\alpha)_{\mathrm{tors}} = H^1(I_v, T)_{\mathrm{tors}}^{\mathrm{Fr}_v=1}$. Since $c_v = \# H^1(I_v, T)_{\mathrm{tors}}^{\mathrm{Fr}_v=1}$, it suffices to prove that

$$\mathrm{im}(\alpha) \cong \mathfrak{O}/\mathrm{ord}_v(\sigma)\mathfrak{O}.$$

The right most vertical isomorphism shows that

$$\mathrm{im}(\alpha) \cong H^1(K_v, \mathfrak{O}(1))/\partial(\mathfrak{O}) \xrightarrow[\mathrm{ord}_v]{\sim} \mathfrak{O}/\mathrm{ord}_v(\sigma)\mathfrak{O}$$

hence Proposition is proved.

□

3.3. **The interpolation and the argument.**

3.3.1. *The case* $\mu = \mathrm{id}$. Let $\mathfrak{p} \subset \mathcal{R}$ be an arithmetic prime and let $f_{\mathfrak{p}}$ be the associated modular form attached to $\mathfrak{p}$. Let $\mathcal{O}(\mathfrak{p}) = \mathcal{R}/\mathfrak{p}$ and $\Phi(\mathfrak{p}) = \mathrm{Frac}\,\mathcal{O}(\mathfrak{p})$ be its field of fractions. When $\mathbb{T}$ is as above, the G_F-representation $\mathbb{T} \otimes \Phi(\mathfrak{p})$ is the Galois representation $V(f_{\mathfrak{p}})$ which is attached to $f_{\mathfrak{p}}$ by Eichler, Shimura and Deligne [Del71]. We define the specialization map

$$s_{\mathfrak{p}} : \mathcal{R} \longrightarrow \mathcal{O}(\mathfrak{p}).$$

Let $S(\mathfrak{p})$ be the integral closure of $\mathcal{O}(\mathfrak{p})$ inside $S(\mathfrak{p})$. By slight abuse, we also write $s_{\mathfrak{p}}$ for the composite

$$s_{\mathfrak{p}} : \mathcal{R} \longrightarrow \mathcal{O}(\mathfrak{p}) \hookrightarrow S(\mathfrak{p}).$$

The ring $S(\mathfrak{p})$ is a DVR. We fix a uniformizer $\pi_\mathfrak{p}$ of $S(\mathfrak{p})$. We write $T = \mathbb{T} \otimes S(\mathfrak{p})$ for the $S(\mathfrak{p})$-lattice inside of $V(f_\mathfrak{p})$. Note that T fits in the exact sequence (3.4) and the arguments of §3.2.2 apply with $\mathfrak{O} = S(\mathfrak{p})$. Let $\tau = \mathrm{ord}_v(\sigma) \in S(\mathfrak{p})$, where σ is as in §3.2.2. We remark here that we also write σ for its image inside $H^1(I_v, S(\mathfrak{p})(1))$ under the isomorphism

$$H^1(K_v, S(\mathfrak{p})(1)) \xrightarrow{\sim} H^1(I_v, S(\mathfrak{p})(1))$$

which appeared in the proof of Proposition 3.3.

Until the end of §3.3.1, suppose the following holds:

Assumption 3.4. $\tau \in S(\mathfrak{p})^\times$.

Remark 3.5. Note that, in view of Proposition 3.3 and the remark just before Proposition 3.3, Assumption 3.4 amounts to say that the Hida family $\mathbb{T}$ should pass through a (twisted) modular form for which the attached Galois representation has the following properties:

(1) p does not divide the the Tamagawa number $\mathrm{Tam}_v(T)$ of the $S(\mathfrak{p})[[G_K]]$-representation T at v,
(2) the exact sequence (3.4) of $S(\mathfrak{p})[[G_v]]$-modules does not split (which is equivalent to saying that T^{I_v} is a free $S(\mathfrak{p})$-module of rank one).

Definition 3.6. Let

$$\boldsymbol{\partial} : \mathcal{R} = \mathcal{R}^{I_v} \longrightarrow H^1(I_v, \mathcal{R}(1))$$

be the connecting homomorphism of the I_v-cohomology of the exact sequence (3.1) (with $\mu = 1$).

Since I_v acts trivially on $\mathcal{R}$, we have

$$H^1(I_v, \mathcal{R}(1)) = H^1(I_v, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathcal{R}.$$

This, together with the isomorphism (3.5) induces

$$H^1(I_v, \mathcal{R}(1)) \xrightarrow{\sim} \mathcal{R} \tag{3.6}$$

which is in fact an isomorphism of $\mathcal{R}[[G_v]]$-modules.

Definition 3.7. Let $\boldsymbol{\tau} \in \mathcal{R}$ be the image of $\boldsymbol{\partial}(1) \in H^1(I_v, \mathcal{R}(1))$ under the isomorphism (3.6). The element $\boldsymbol{\tau}$ is called to *Tamagawa element* attached to the family $\mathbb{T}$.

Proposition 3.8. *Suppose Assumption 3.4 holds, then* $\boldsymbol{\tau} \in \mathcal{R}^\times$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
1 \arrow[d, mapsto] \arrow[rrrrrr, bend left, mapsto] & \in & \mathcal{R} \arrow[r, "\boldsymbol{\partial}"] \arrow[d, "s_{\mathfrak{p}}"] & H^1(I_v,\mathcal{R}(1)) \arrow[r, "\sim"] \arrow[d, "s_{\mathfrak{p}}"] & \mathcal{R} \arrow[d, "s_{\mathfrak{p}}"] & \ni & \boldsymbol{\tau} \arrow[d, mapsto] \\
1 \arrow[rrrrrr, bend right, mapsto] & \in & S(\mathfrak{p}) \arrow[r] & H^1(I_v,S(\mathfrak{p})(1)) \arrow[r, "\sim"] & S(\mathfrak{p}) & \ni & \tau
\end{tikzcd}$$

Let $\bar{\tau}$ is the image of τ under the map

$$S(\mathfrak{p}) \longrightarrow k(\mathfrak{p}) := S(\mathfrak{p})/\pi_{\mathfrak{p}}S(\mathfrak{p}).$$

By our assumption that $\tau \in S(\mathfrak{p})^\times$, it follows that $\bar{\tau} \neq 0$. Furthermore, there is a commutative diagram

$$\begin{tikzcd}
\boldsymbol{\tau} \arrow[drrrrrrr, bend right, mapsto] & \in & \mathcal{R} \arrow[rr, "s_{\mathfrak{p}}"] \arrow[dr] & & \mathcal{O}(\mathfrak{p}) \arrow[rr, hook] & & S(\mathfrak{p}) \arrow[dl] & & \\
& & & \mathcal{R}/\mathfrak{m} \arrow[rr, hook] & & k(\mathfrak{p}) & \ni & \bar{\tau} &
\end{tikzcd}$$

where the injection $\mathcal{R}/\mathfrak{m} \hookrightarrow k(\mathfrak{p})$ is because $\mathcal{R}/\mathfrak{m}$ is the unique field that $\mathcal{R}$ surjects onto. This shows that the image of τ under the natural map $\mathcal{R} \to \mathcal{R}/\mathfrak{m}$ is non-zero, and Proposition follows. □

Let now

$$s : \mathcal{R} \longrightarrow S$$

be any specialization, where S is a discrete valuation ring with uniformizer π_S. We set $T_S = \mathbb{T} \otimes_{\mathcal{R}} S$. Let $\mathrm{Tam}_v(T_S)$ denote the Tamagawa number at v for the $S[[G_F]]$-representation T_S. Note that T_S is a free S-module of rank two and we use the definition of [FPR94] of Tamagawa factors.

Proposition 3.9. *Suppose that Assumption 3.4 above holds true. Then* $\mathrm{Tam}_v(T_S)$ *is prime to* p.

Proof. As in the proof of Proposition 3.8, there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[d, mapsto] \arrow[rrrrrr, bend left, mapsto] & \in & \mathcal{R} \arrow[r, "\boldsymbol{\partial}"] \arrow[d, "s"] & H^1(I_v,\mathcal{R}(1)) \arrow[r, "\sim"] \arrow[d, "s"] & \mathcal{R} \arrow[d, "s"] & \ni & \boldsymbol{\tau} \arrow[d, mapsto] \\
1 \arrow[rrrrrr, bend right, mapsto] & \in & S \arrow[r] & H^1(I_v,S(1)) \arrow[r, "\sim"] & S & \ni & \tau_S
\end{tikzcd}$$

Also, the commutative diagram

$$\begin{tikzcd}
\boldsymbol{\tau} \arrow[d, mapsto] \arrow[drrrrr, bend left, mapsto] & \in & \mathcal{R} \arrow[r, "s"] \arrow[d] & S \arrow[d] & & \\
\bar{\boldsymbol{\tau}} \arrow[rrrrr, bend right, mapsto] & \in & \mathcal{R}/\mathfrak{m} \arrow[r, hook] & S/\pi_S S & \ni & \overline{\tau_S}
\end{tikzcd}$$

shows that $\tau_S \in S^\times$, since $\bar{\boldsymbol{\tau}} \neq 0$ by Proposition 3.8. This completes the proof by making use of Proposition 3.3 with $\mathfrak{O} = S$ and $T = T_S$. $\square$

3.3.2. *The case* $\mu \neq \mathrm{id}$. The case $\mu \neq$ id (in (3.1)) is handled just as in §3.2.1. Following the proof of Proposition 3.2, one is able to prove:

Proposition 3.10. *Suppose T_S is as in §3.3.1. Then under Assumption 3.4, $p \nmid \mathrm{Tam}_v(T_S)$.*

The following Lemma will be crucial when checking the local properties of the big Heegner point Euler system.

Lemma 3.11. *Let $\mathfrak{L}$ be any unramified p-extension of K_v. Then under Assumption 3.4, the $\mathcal{R}$-module $H^1(\mathfrak{L}^{\mathrm{ur}}, \mathbb{T})$ is torsion-free.*

Proof. As the Assumption ($\mathrm{H}_{\mathbf{v}}$) is in effect, the G_v-representation $\mathbb{T}$ fits in an exact sequence

$$0 \longrightarrow \mathcal{R}(1) \otimes \mu \longrightarrow \mathbb{T} \longrightarrow R \longrightarrow 0,$$

where the character μ is described above.

In the case $\mu \neq$ id, the proof of Proposition 3.2 shows that $\mathbb{T} = (\mathcal{R}(1) \otimes \mu) \oplus \mathcal{R}$ as G_v-modules. Thus, we need to verify that the $\mathcal{R}$-modules $H^1(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R}(1) \otimes \mu)$ and $H^1(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R})$ are torsion-free. For the module

$$H^1(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R}) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}_v/\mathfrak{L}^{\mathrm{ur}}), \mathcal{R})$$

this is clear. Also, when μ is an unramified character of G_v, it follows that the inertia acts trivially on $\mathcal{R}(1) \otimes \mu$ and hence

$$H^1(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R}(1) \otimes \mu) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}_v/\mathfrak{L}^{\mathrm{ur}}), \mathcal{R}(1) \otimes \mu) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}_v/\mathfrak{L}^{\mathrm{ur}}), \mathcal{R})$$

is free as well. When μ is ramified, the argument carries over after replacing $\mathfrak{L}^{\mathrm{ur}}$ by a quadratic extension.

In the case $\mu = \mathrm{id}$, we have the following exact sequence as in the proof of Proposition 3.3:

$$\mathcal{R} = H^0(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R}) \xrightarrow{\partial} H^1(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R}(1)) \xrightarrow{\alpha} H^1(\mathfrak{L}^{\mathrm{ur}}, \mathbb{T}) \longrightarrow H^1(\mathfrak{L}^{\mathrm{ur}}, \mathcal{R}) \cong \mathcal{R}$$

Since $\mathfrak{L}/K_v$ is unramified, the proof of Proposition 3.8 shows that the map ∂ is surjective under Assumption (3.4), hence the map α is injective. The proof is now complete. $\square$

4. Kolyvagin descent for the big Heegner points

4.1. **Selmer groups.**

4.1.1. *Local conditions and Selmer structures.* Throughout this section, let L be a finite extension of K and for each prime v of L define L_v^{unr} as the maximal unramified extension of L_v. Let $\mathcal{I}_v \subset \mathcal{D}_v$ be a fixed choice of inertia and decomposition groups of v. Let R be any local Noetherian ring and M any $R[[G_K]]$-module.

Definition 4.1. A *Selmer structure* $\mathcal{F}$ on M is a collection of the following data:

- For every $v \mid Np$, choose a local condition on M (which we view now as a $R[[\mathcal{D}_v]]$-module), i.e., a choice of R-submodule
$$H^1_{\mathcal{F}}(L_v, M) \subset H^1(L_v, M).$$
- For $v \nmid Np$, set
$$H^1_{\mathcal{F}}(L_v, M) = H^1_f(L_v, M) := \ker\left(H^1(L_v, M) \to H^1(L_v^{\mathrm{unr}}, M)\right)$$

Definition 4.2. The *semi-local cohomology group* at a rational prime ℓ is defined by setting

$$H^i(L_\ell, M) := \bigoplus_{v \mid \ell} H^i(L_v, M),$$

where the direct sum is over all primes v of L lying above ℓ.

Definition 4.3. If $\mathcal{F}$ is a Selmer structure on M, we define the *Selmer module* $H^1_{\mathcal{F}}(L, M)$ as

$$H^1_{\mathcal{F}}(L, M) := \ker\left(H^1(L, M) \longrightarrow \prod_v H^1(L_v, M)/H^1_{\mathcal{F}}(L_v, M)\right).$$

4.1.2. *Greenberg Conditions.* Let $\mathbb{T}$ be the big self-dual Galois representation attached to a Hida family.

Proposition 4.4. *Suppose v is any place of L above p. Then there is an exact sequence of $\mathcal{R}[[D_v]]$-modules*

$$0 \longrightarrow \mathrm{F}_v^+(\mathbb{T}) \longrightarrow \mathbb{T} \longrightarrow \mathrm{F}_v^-(\mathbb{T}) \longrightarrow 0 \tag{4.1}$$

such that both $\mathrm{F}_v^+(\mathbb{T})$ and $\mathrm{F}_v^-(\mathbb{T})$ are free of rank one over $\mathcal{R}$.

See [How07, Prop. 2.4.1] (and [Nek06, §12.7.8-10] when dealing with a Hida family of Hilbert modular forms) for a proof of this statement.

For any ring homomorphism $\mathfrak{s} : \mathcal{R} \to S$ (where S is a local Noetherian ring), set $T_S = \mathbb{T} \otimes_{\mathfrak{s}} S$. By tensoring the sequence (4.1) by S, we also define $\mathrm{F}_v^{\pm}(T_S)$ for any of the modules T_S above. Furthermore, if S is a ring which is finitely generated as an $\mathcal{O}$-module, set $V_S = T_S \otimes_{\mathcal{O}} E$ and define $\mathrm{F}_v^{\pm}(V_S)$ in a similar manner.

Definition 4.5.

(i) The *strict Greenberg Selmer structure* $\mathcal{F}_{\mathrm{Gr}}$ on $\mathbb{T}$ by setting local conditions as

$$H^1_{\mathcal{F}_{\mathrm{Gr}}}(L_v, \mathbb{T}) = \begin{cases} \ker\left(H^1(L_v, \mathbb{T}) \longrightarrow H^1(L_v^{\mathrm{unr}}, \mathbb{T})\right) & , \quad \text{if } v \nmid p \\ \ker\Big(H^1(L_v, \mathbb{T}) \longrightarrow H^1(L_v, \mathrm{F}^-(\mathbb{T}))\Big) & , \quad \text{if } v \mid p \end{cases}$$

(ii) Let T be any subquotient of $\mathbb{T}$. Then let $\mathcal{F}_{\mathrm{Gr}}$ on T be the Selmer structure defined by propagating the Selmer structure $\mathcal{F}_{\mathrm{Gr}}$ on $\mathbb{T}$ via [MR04, Example 1.1.2].

(iii) Let S be a ring for which T_S and V_S is defined. We define a Selmer structure $\tilde{\mathcal{F}}_{\mathrm{Gr}}$ on T_S by setting

$$H^1_{\tilde{\mathcal{F}}_{\mathrm{Gr}}}(L_v, T_S) = \begin{cases} \ker\left(H^1(L_v, T_S) \to H^1(L_v^{\mathrm{unr}}, V_S)\right) & , \quad \text{if } v \mid N \\ H^1_{\mathcal{F}_{\mathrm{Gr}}}(L_v, T_S) & , \quad \text{if } v \nmid N \end{cases}$$

Proposition 4.6. *Suppose the Assumption 3.1 and Assumption 3.4 holds; see Remark 3.5 for the content of the latter assumption. Then*

$$H^1_{\tilde{\mathcal{F}}_{\mathrm{Gr}}}(L_v, T_S) = H^1_{\mathcal{F}_{\mathrm{Gr}}}(L_v, T_S)$$

for all v.

Proof. This follows from Proposition 3.9 and [Rub00, Lemma 3.5]. $\square$

4.2. **Kolyvagin systems for $\mathbb{T}$ and $\mathbb{T}\otimes\Lambda^{\mathrm{ac}}$.** Let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K and $\Gamma^{\mathrm{ac}} = \mathrm{Gal}(K_\infty/K)$, $\Lambda^{\mathrm{ac}} = \mathbb{Z}_p[[\Gamma^{\mathrm{ac}}]]$. Let $\gamma \in \Gamma^{\mathrm{ac}}$ be a fixed topological generator, and let $\{\pi, x\}$ be a maximal regular sequence for the two-dimensional Gorenstein ring $\mathcal{R}$, where $\pi = \pi_E$ is the uniformizer of E fixed in the introduction. For each $k, m, r \in \mathbb{Z}^+$, let $R_{k,m}$ (resp., $R_{k,m,r}$) be the artinian local ring $\mathcal{R}/(\pi^k, x^m)$ (resp., $R_{k,m}\otimes_{\mathbb{Z}_p} \Lambda^{\mathrm{ac}}/(\gamma-1)^r$) and $T_{k,m}$ (resp., $T_{k,m,r}$) be the module $\mathbb{T}\otimes_{\mathcal{R}} R_{k,m}$ (resp., $\mathbb{T}\otimes_{\mathcal{R}} R_{k,m,r}$). Note that we allow G_K act on both factors defining $T_{k,m,r}$ via the map $G_K \twoheadrightarrow \Gamma^{\mathrm{ac}}$. Let $\mathcal{P}$ be the set of degree two primes of K which contains no prime that divides Np and define $\mathcal{P}_{k,m,r} \in \mathcal{P}$ to be the collection of primes λ which satisfy:

(i) Let ℓ be the rational prime below λ. Then $\ell + 1 \equiv 0 \mod p^k$.

(ii) Let $D_\lambda \subset G_K$ be any decomposition group for the prime λ. Then
$$D_\lambda \subset \ker(G_K \to \mathrm{Aut}_{\mathcal{R}}(T_{k,m,r})).$$
Note that this condition is independent of the choice of the decomposition group D_λ and implies that Fr_λ acts trivially on $T_{k,m,r}$.

For $\lambda \in \mathcal{P}_{k,m,r}$, set
$$H^1_f(K_\lambda, T_{k,m,r}) := \ker(H^1(K_\lambda, T_{k,m,r}) \longrightarrow H^1(K_\lambda^{\mathrm{unr}}, T_{k,m,r})),$$
and
$$H^1_s(K_\lambda, T_{k,m,r}) := H^1(K_\lambda, T_{k,m,r})/H^1_f(K_\lambda, T_{k,m,r}).$$

Proposition 4.7. *Suppose $\lambda \in \mathcal{P}_{k,m,r}$. Let $\mathrm{k}_\lambda = \mathcal{O}_K/\lambda$ and $\mathrm{k}_\ell = \mathbb{Z}/\ell\mathbb{Z}$. Then there is a finite-singular comparison map*
$$\phi_\lambda^{\mathrm{fs}} : H^1_f(K_\lambda, T_{k,m,r}) \xrightarrow{\sim} H^1_s(K_\lambda, T_{k,m,r}) \otimes \mathrm{k}_\lambda^\times/\mathrm{k}_\ell^\times.$$

Proof. It follows from [MR04, Lemma 1.2.1] and our assumption $\lambda \in \mathcal{P}_{k,m,r}$ that
$$H^1_f(K_\lambda, T_{k,m,r}) \xrightarrow{\sim} T_{k.m,n} \xleftarrow{\sim} H^1_s(K_\lambda, T_{k,m,r}) \otimes \mathrm{k}_\lambda^\times.$$
Identifying the p-Sylow subgroups of $\mathrm{k}_\lambda^\times$ and $\mathrm{k}_\lambda^\times/\mathrm{k}_\ell^\times$ the Proposition follows. □

Definition 4.8.

(a) For $\lambda \in \mathcal{P}$ and ℓ the prime below λ, define $\mathcal{G}(\ell) = \mathrm{k}_\lambda^\times/\mathrm{k}_\ell^\times$.

(b) Let $\mathcal{N}_{k,m,r}$ (resp., $\mathcal{N}$) be the set of square-free products of the rational primes ℓ that lie above the primes chosen among of $\mathcal{P}_{k,m,r}$ (resp., $\mathcal{P}$).

(c) For $n \in \mathcal{N}$, define $\mathcal{G}(n) = \bigotimes_{\ell | n} \mathcal{G}(\ell)$.

Definition 4.9. For $\lambda \in \mathcal{P}$ and ℓ the prime below it, let H_ℓ be the ring class field of conductor ℓ. Since λ splits completely in the Hilbert class field of K, the maximal p-subextension L of the extension $(H_\ell)_\lambda/K_\lambda$ is totally ramified abelian p-extension of K_λ. Furthermore, its Galois group is canonically identified with the p-Sylow subgroup of G_λ by class field theory hence it is also the maximal totally tamely ramified abelian p-extension of K_λ. Define the *transverse submodule*

$$H^1_{\text{tr}}(K_\lambda, M) = \ker(H^1(K_\lambda, M) \longrightarrow H^1(L, M))$$

for any G_{K_λ}-module M.

Lemma 4.10. *The transverse submodule $H^1_{\text{tr}}(K_\lambda, T_{k,m,r})$ projects isomorphically onto the singular quotient $H^1_s(K_\lambda, T_{k,m,r})$ under the natural projection*

$$H^1(K_\lambda, T_{k,m,r}) \longrightarrow H^1_s(K_\lambda, T_{k,m,r}).$$

This is [MR04, Lemma 1.2.4].

Definition 4.11. For a Selmer structure $\mathcal{F}$ on a G_K-representation M and $n \in \mathcal{N}$, define the modified Selmer structure $\mathcal{F}(n)$ on M by setting

$$H^1_{\mathcal{F}(n)}(K_v, M) = \begin{cases} H^1_{\mathcal{F}}(K_v, M) & , \quad \text{if } v \nmid n \\ H^1_{\text{tr}}(K_v, M) & , \quad \text{if } v \mid n \end{cases}$$

Definition 4.12. The $\mathcal{R} \otimes \Lambda^{\text{ac}}$-module of big Kolyvagin systems for $\mathbb{T} \otimes \Lambda^{\text{ac}}$ is defined as

$$\overline{\mathbf{KS}}(\mathbb{T} \otimes \Lambda^{\text{ac}}, \mathcal{F}_{\text{Gr}}) := \varprojlim \mathbf{KS}(T_{k,m,r}, \mathcal{F}_{\text{Gr}}, \mathcal{P}_{k,m,r}),$$

where each of the modules $\mathbf{KS}(T_{k,m,r}, \mathcal{F}_{\text{Gr}}, \mathcal{P}_{k,m,r})$ of Kolyvagin systems over the artinian ring $R_{k,m,r}$ is defined following [MR04, Definition 3.1.3], via the constructions given above.

4.3. **Big Heegner point Kolyvagin system.** We start this section by recalling Kolyvagin's derivative construction. Let H_c denote the ring class field of K of conductor c, and for c prime to p, let $K(c)$ be the maximal p-extension in H_c/K. We assume until the end that

- The class number of K is prime to p (equivalently, $K(1) = K$).
- $\overline{T}$ is an absolutely irreducible G_K-representation.
- The twisted Hida family passes through a member for which all the Tamagawa factors at primes ℓ dividing the tame conductor N are prime to p.

Then the maximal p-extension K_α in $H_{p^{\alpha+1}}$ satisfies that $[K_\alpha : K] = p^\alpha$. For $(c,p)=1$, write $K_\alpha(c)$ for the composite field of $K(c)$ and K_α. For Howard's big Euler system $\{\mathfrak{X}_{cp^\alpha}\}_{c,\alpha}$ of Heegner points defined as in [How07, §2.2], we set using [How07, Prop. 2.3.1]

$$\mathfrak{z}_{c,\alpha} := \mathrm{cor}_{H_{cp^{\alpha+1}}/K_\alpha(c)} U_p^{-\alpha} \mathfrak{X}_{cp^{\alpha+1}} \in H^1(K_\alpha(c), \mathbb{T})$$

for every c prime to Np, where U_p is the Hecke operator.

Remark 4.13. Let λ a place of K and suppose L/K_λ is an unramified extension. Then $L^{\mathrm{ur}} = K_\lambda^{\mathrm{ur}}$, as the field L^{ur} is unramified over K_λ, and the composite $K_\lambda^{\mathrm{ur}} L$ is unramified over L.

Proposition 4.14. *Under the running hypotheses,*

$$\mathfrak{z}_{c,\alpha} \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\alpha(c), \mathbb{T}).$$

Proof. We need to check that $\mathfrak{z}_{c,\alpha} \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\alpha(c)_v, \mathbb{T})$ for every place v of $K_\alpha(c)$.

Suppose that $w|v|\lambda|N$, where w (resp., v, resp., λ) is a prime of $H_{cp^{\alpha+1}}$ (resp., of $K_\alpha(c)$, resp., of K). For ease of notation, let $\mathfrak{L} = (H_{cp^{\alpha+1}})_w$ and $\mathfrak{K} = K_\alpha(c)_v$. The proof of [How07, Prop. 2.4.5] shows that the restriction of $\mathfrak{X}_{cp^\alpha}$ to $H^1(\mathfrak{L}^{\mathrm{ur}}, \mathbb{T})$ is $\mathcal{R}$-torsion, which is trivial by Lemma 3.11. This completes the proof that $\mathrm{loc}_w(\mathfrak{X}_{cp^\alpha}) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(\mathfrak{L}, \mathbb{T})$. We have a commutative diagram

$$\begin{tikzcd}
H^1(\mathfrak{L}, \mathbb{T}) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{cores}"] & H^1(I_\lambda, \mathbb{T})^{G_{\mathfrak{L}}} \arrow[d, "\mathrm{N}_{\mathfrak{K}/\mathfrak{L}}"] \\
H^1(\mathfrak{K}, \mathbb{T}) \arrow[r, "\mathrm{res}"] & H^1(I_\lambda, \mathbb{T})^{G_{\mathfrak{K}}}
\end{tikzcd}$$

where we use Remark 4.13 to identify I_λ with the Galois groups of the extensions $\overline{K}_\lambda/\mathfrak{K}^{\mathrm{ur}}$ and $\overline{K}_\lambda/\mathfrak{L}^{\mathrm{ur}}$. Since the image of $\mathrm{loc}_w(\mathfrak{X}_{cp^\alpha})$ under the left vertical map is $\mathrm{loc}_v(\mathfrak{z}_{c,\alpha})$, and its image under the upper horizontal map is trivial, it follows that $\mathrm{loc}_v(\mathfrak{z}_{c,\alpha}) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(\mathfrak{K}, \mathbb{T})$ as desired.

For a prime $w \nmid Np$ of $H_{cp^{\alpha+1}}$, Howard in [How07, Prop. 2.4.5] proves that $\mathrm{loc}_w(\mathfrak{X}_{cp^\alpha}) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(\mathfrak{L}, \mathbb{T})$ and the proposition follows for every $v \nmid N$ as above.

Finally, for $w \mid p$ of $H_{cp^{\alpha+1}}$, Howard in loc. cit. shows that $\mathrm{loc}_w(\mathfrak{X}_{cp^\alpha}) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(\mathfrak{L}, \mathbb{T})$ and the proposition follows from the commutative diagram:

$$\begin{tikzcd}
H^1(\mathfrak{L}, \mathbb{T}) \arrow[r] \arrow[d] & H^1(\mathfrak{L}, \mathrm{F}_v^-(\mathbb{T})) \arrow[d] \\
H^1(\mathfrak{K}, \mathbb{T}) \arrow[r] & H^1(\mathfrak{K}, \mathrm{F}_v^-(\mathbb{T}))
\end{tikzcd}$$

□

Let
$$\mathcal{A}_s = \ker\left(\mathcal{O}[[\mathrm{Gal}(H_{p^\infty}/H_{p^s})]] \longrightarrow \mathcal{O}\right)$$
be the augmentation ideal of $\mathcal{O}[[\mathrm{Gal}(H_{p^\infty}/H_{p^s})]]$. Until the end of this section fix $k, m, r \in \mathbb{Z}^+$ as well as a positive integer s for which the following condition holds:

(4.2) The image of $\mathcal{A}_s$ is contained in the ideal $(\pi^k, (\gamma-1)^{p^r})$ of Λ^{ac} under the map induced from the natural inclusion $\mathrm{Gal}(H_{p^\infty}/H_{p^s}) \subset \Gamma^{\mathrm{ac}}$.

Definition 4.15. Fix a prime $\lambda \in \mathcal{P}_{k,m,r}$ and let ℓ be the rational prime below λ.

(i) Let $\mathcal{G}_\ell = \mathrm{Gal}(K(\ell)/K)$. Note then that $\mathcal{G}_\ell$ is the p-Sylow subgroup of the cyclic group $\mathcal{G}(\ell) = k_\lambda^\times/k_\ell^\times$ defined above. Let σ_ℓ be the generator of $\mathcal{G}_\ell$.

(ii) For a square free integer $n \in \mathcal{N}_{k,m,r}$, we set $\mathcal{G}_n = \bigotimes_{\ell|n} \mathcal{G}_\ell$ and define $G(n) = \prod_{\ell|n} \mathcal{G}_\ell$. Then for $m|n$,
$$\mathrm{Gal}(K(n)/K(m)) \cong \prod_{\ell|n} \mathcal{G}_\ell \cong G(n/m).$$
Since we assumed that p is prime to the class number of K, we also have that
$$G(n) \cong \mathrm{Gal}(K(n)/K).$$

(iii) $\mathcal{D}_\ell = \displaystyle\sum_{i=0}^{|\mathcal{G}_\ell|-1} i\sigma_\ell^i \in \mathbb{Z}_p[\mathcal{G}_\ell]$ and $D_n = \displaystyle\prod_{\ell|n} D_\ell \in \mathbb{Z}_p[G(n)]$.

Definition 4.16. For $n \in \mathcal{N}_{k,m,r}$ and $\alpha \geq s$, define $\mathfrak{z}'_{n,\alpha} = D_n\,\mathfrak{z}_{n,\alpha} \in H^1(K_\alpha(n), \mathbb{T})$.

By the standard telescoping identity satisfied by the derivative operators D_n, it follows for $n \in \mathcal{N}_{k,m,r}$ that the image of $\mathfrak{z}'_{n,\alpha}$ (which we denote by $\kappa'_{[n,\alpha]}$) under the reduction map
$$H^1(K_\alpha(n), \mathbb{T}) \longrightarrow H^1(K_\alpha(n), T_{k,m})$$
lies inside $H^1(K_\alpha(n), T_{k,m})^{G(n)}$. On the other hand, since $G(n)$ is generated by the (p-parts of the) inertia groups at the primes dividing n and $T_{k,m}$ is unramified at these primes, it follows that
$$H^0(K(n), T_{k,m}) = H^0(K, T_{k,m}).$$

Furthermore, since we assumed that the G_K-representation $\overline{T}$ is irreducible, we have that $H^0(K, T_{k,m}) = 0$. The restriction map

$$H^1(K_\alpha, T_{k,m}) \longrightarrow H^1(K_\alpha(n), T_{k,m})^{G(n)} \tag{4.3}$$

is therefore an isomorphism.

Definition 4.17.

(i) For $n \in \mathcal{N}_{k,m,r}$ and $\alpha \geq s$, define $\kappa_{[n,\alpha]}$ as the inverse image of $\kappa'_{[n,\alpha]}$ under the isomorphism (4.3).

(ii) Let $\kappa_n \in H^1(K, T_{k,m,r})$ be the image of $\kappa_{[n,\alpha]}$ under the map

$$H^1(K_\alpha, T_{k,m}) \longrightarrow H^1(K, T_{k,m,r})$$

induced by Shapiro's Lemma, and the choice of s as in (4.2). It is not hard to see that the definition of κ_n does not depend on the choice of α and s, thanks to the norm compatibility of the classes $\mathfrak{z}_{c,\alpha}$ as α varies.

Recall that $\mathfrak{z}_{n,\alpha} \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\alpha(n), \mathbb{T})$ by Proposition 4.14.

4.3.1. *Local properties away from Np.*

Proposition 4.18. *For a place $v \nmid Nnp$ of K and a place w of K_α above v, we have*

(i) $\mathrm{loc}_w(\kappa_{[n,\alpha]}) \in H^1_{\mathrm{ur}}(K_\alpha)_w, T_{k,m}) := \ker(H^1((K_\alpha)_w, T_{k,m}) \longrightarrow H^1((K_\alpha)^{\mathrm{ur}}_w, T_{k,m}))$,

(ii) $\mathrm{loc}_v(\kappa_n) \in H^1_{\mathrm{ur}}(K_v, T_{k,m,r}) := \ker(H^1(K_v, T_{k,m,r}) \longrightarrow H^1(K^{\mathrm{ur}}_v, T_{k,m,r}))$.

Proof. Let w' be a place of $K_\alpha(n)$ above w. As remarked above, we have $K_\alpha(n)^{\mathrm{ur}}_{w'} = (K_\alpha)^{\mathrm{ur}}_w = K^{\mathrm{ur}}_v$. Using the fact that $\mathrm{loc}_{w'}(\mathfrak{z}_{n,\alpha})$ (and therefore $\mathrm{loc}_{w'}(\mathfrak{z}'_{n,\alpha})$ as well) lies in

$$H^1_{\mathrm{ur}}(K_\alpha(n)_{w'}, \mathbb{T}) = \ker(H^1(K_\alpha(n)_{w'}, \mathbb{T}) \longrightarrow H^1(K^{\mathrm{ur}}_v, \mathbb{T})),$$

the diagram below with commutative squares proves (i):

$$\begin{tikzcd}
H^1((K_\alpha)_w, T_{k,m}) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & H^1(K^{\mathrm{ur}}_v, T_{k,m}) \arrow[d, "\mathrm{id}"] \\
H^1(K_\alpha(n)_{w'}, T_{k,m}) \arrow[r, "\mathrm{res}"] & H^1(K^{\mathrm{ur}}_v, T_{k,m}) \\
H^1(K_\alpha(n)_{w'}, \mathbb{T}) \arrow[r, "\mathrm{res}"] \arrow[u] & H^1(K^{\mathrm{ur}}_v, \mathbb{T}) \arrow[u]
\end{tikzcd}$$

Semi-local Shapiro's Lemma yields the upper square of the following commutative diagram:

$$\begin{tikzcd}
\oplus_{w|v} H^1(K_v^{\mathrm{ur}}/(K_\alpha)_w, T_{k,m}) \arrow[r] \arrow[d, "\cong"] & \oplus_{w|v} H^1((K_\alpha)_w, T_{k,m}) \arrow[d, "\cong"] \\
H^1(K_v^{\mathrm{ur}}/K_v, \mathrm{Ind}_{K_\alpha/K}\, T_{k,m}) \arrow[r] \arrow[d] & H^1(K_v, \mathrm{Ind}_{K_\alpha/K}\, T_{k,m}) \arrow[d] \\
H^1(K_v^{\mathrm{ur}}/K_v, T_{k,m,r}) \arrow[r] & H^1(K_v, T_{k,m,r})
\end{tikzcd}$$

(i) shows that $\{\mathrm{loc}_w(\kappa_{[n,\alpha]})\}_{w|v}$ is in the image of the uppermost horizontal arrow, which implies that $\mathrm{loc}_v(\kappa_n)$ is in the image of the lowermost horizontal arrow. This completes the proof. $\square$

Proposition 4.19. *For $\lambda|n$, we have $\kappa_n \in H^1_{\mathrm{tr}}(K_\lambda, T_{k,m,r})$.*

Proof. This is standard, c.f., Lemma 1.7.3 and 2.3.4 of [How04]. $\square$

4.3.2. *Local properties at p.* Let v be a place of K above p.

Proposition 4.20. *For any place w of K_α above v, we have*

$$\mathrm{loc}_w(\kappa_{[n,\alpha]}) \in \ker\left(H^1((K_\alpha)_w, T_{k,m}) \longrightarrow H^1((K_\alpha)_w, \mathrm{F}_v^-(T_{k,m}))\right)$$

Proof. Let $a \geq \alpha$ be a positive integer. The fact that $\mathfrak{z}_{n,a} \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_a(n), \mathbb{T})$ and the $G(n)$-equivariance of the map

$$H^1(K_a(n)_v, \mathbb{T}) := \bigoplus_{\wp|v} H^1(K_a(n)_\wp, \mathbb{T}) \longrightarrow \bigoplus_{\wp|v} H^1(K_a(n)_\wp, \mathrm{F}_v^-(\mathbb{T})) = H^1(K_a(n)_v, \mathrm{F}_v^-(\mathbb{T}))$$

shows that

$$\mathrm{loc}_v(\mathfrak{z}'_{n,a}) \in \ker\left(H^1(K_a(n)_v, \mathbb{T}) \longrightarrow H^1(K_a(n)_v, \mathrm{F}_v^-(\mathbb{T})\right).$$

This, along with the commutative diagram

$$\begin{tikzcd}
H^1(K_a(n)_v, \mathbb{T}) \arrow[r] \arrow[d] & H^1(K_a(n)_v, , \mathrm{F}_v^-(\mathbb{T})) \arrow[d] \\
H^1(K_a(n)_v, T_{k,m}) \arrow[r] & H^1(K_a(n)_v, , \mathrm{F}_v^-(T_{k,m}))
\end{tikzcd}$$

proves that $\mathrm{loc}_v(\kappa'_{[n,a]}) \in \ker\left(H^1(K_a(n)_v, T_{k,m}) \to H^1(K_a(n)_v, \mathrm{F}_v^-(T_{k,m})\right)$, and hence that

$$\mathrm{loc}_w(\kappa_{[n,a]}) \in \ker\left(H^1((K_a)_w, T_{k,m}) \longrightarrow H^1(K_a(n)_{w'}, \mathrm{F}_v^-(T_{k,m})\right)$$

for every prime w of K_a above v and w' of $K_a(n)$ above w. Let $c_{[n,a]}$ be the image of $\mathrm{loc}_w(\kappa_{[n,a]})$ under

$$H^1((K_a)_w, T_{k,m}) \longrightarrow H^1((K_a)_w, \mathrm{F}_v^-(T_{k,m})).$$

We wish to prove that $c_{[n,\alpha]} = 0$. The map

$$H^1((K_a)_w, T_{k,m}) \longrightarrow H^1(K_a(n)_{w'}, \mathrm{F}_v^-(T_{k,m}))$$

factors as

$$\begin{tikzcd} H^1((K_a)_w, T_{k,m}) \arrow[rr] \arrow[dr] & & H^1(K_a(n)_{w'}, \mathrm{F}_v^-(T_{k,m})) \\ & H^1((K_a)_w, \mathrm{F}_v^-(T_{k,m})) \arrow[ur, "\rho_a"'] & \end{tikzcd}$$

which in return shows that $c_{[n,a]} \in \ker(\rho_a)$. Using the norm compatibility of the classes $\kappa_{[n,a]}$ (and hence that of $c_{[n,a]}$) as a varies, it follows that $c_{[n,\alpha]} = \mathrm{cor}_{K_a(n)_{w'}/(K_a)_w}(c_{[n,a]})$ and thus it suffices to show that $\varprojlim_a \ker(\rho_a) = 0$ where the inverse limit is with respect to the corestriction maps. By inflation-restriction

$$\ker(\rho_a) \cong H^1\left(K_a(n)_{w'}/(K_a)_w, H^0(K_a(n)_{w'}, \mathrm{F}_v^-(T_{k,m}))\right),$$

and we are therefore reduced to checking the vanishing

$$\varprojlim_a H^0(K_a(n)_{w'}, \mathrm{F}_v^-(T_{k,m})) = 0.$$

But this is clear, as the size of the modules $H^0(K_a(n)_{w'}, \mathrm{F}_v^-(T_{k,m}))$ are bounded independently of a, hence these modules stabilize for large enough a and the corestriction maps then are multiplication by powers of p. $\square$

Remark 4.21. Proposition 4.20 is equivalent to saying that

$$\mathrm{loc}_w(\kappa_{[n,\alpha]}) \in \mathrm{im}\left(H^1((K_\alpha)_w, , \mathrm{F}_v^+(T_{k,m})) \longrightarrow H^1((K_\alpha)_w, T_{k,m})\right).$$

Corollary 4.22. $\mathrm{loc}_v(\kappa_n) \in \mathrm{im}\left(H^1(K_v, \mathrm{F}_v^+(T_{k,m,r})) \longrightarrow H^1(K_v, T_{k,m,r})\right).$

Proof. This follows at once from Remark 4.21 and the following commutative diagram which we obtain using Shapiro's Lemma:

$$\begin{tikzcd} H^1((K_\alpha)_w, \mathrm{F}_v^+(T_{k,m})) \arrow[r] \arrow[d] & H^1((K_\alpha)_w, T_{k,m}) \arrow[d] \\ H^1(K_v, \mathrm{F}_v^+(T_{k,m,r})) \arrow[r] & H^1(K_v, T_{k,m,r}) \end{tikzcd}$$

$\square$

Note that Corollary 4.22 alone is not enough to conclude that

$$\mathrm{loc}_v(\kappa_n) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_v, T_{k,m,r}) := \mathrm{im}\left(H^1(K_v, \mathrm{F}_v^+(\mathbb{T}\otimes\Lambda^{\mathrm{ac}})) \longrightarrow H^1(K_v, T_{k,m,r})\right).$$

Consider the following hypothesis, which may be thought of as a condition to avoid *trivial zeros* at characters of Γ^{ac} of finite order:

(**H.stz**) $H^0(K_v, \mathrm{F}_v^-(\overline{T})) = 0$.

We assume until the end that **H.stz** holds. It follows by local duality and the fact that the cohomological dimension of G_v is 2 (and using Nakayama's Lemma) that

$$H^2(K_v, \mathrm{F}_v^+(\mathbb{T}\otimes\Lambda^{\mathrm{ac}})) = 0,$$

and hence we have a surjection

$$H^1(K_v, \mathrm{F}_v^+(\mathbb{T}\otimes\Lambda^{\mathrm{ac}}) \twoheadrightarrow H^1(K_v, \mathrm{F}_v^+(T_{k,m,r})).$$

This shows that

$$\mathrm{im}\left(H^1(K_v, \mathrm{F}_v^+(\mathbb{T}\otimes\Lambda^{\mathrm{ac}})) \to H^1(K_v, T_{k,m,r})\right) = \mathrm{im}\left(H^1(K_v, \mathrm{F}_v^+(T_{k,m,r})) \to H^1(K_v, T_{k,m,r})\right)$$

and thus

Corollary 4.23. *If one assumes* **H.stz** *then* $\mathrm{loc}_v(\kappa_n) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_v, T_{k,m,r})$.

Remark 4.24. The reader familiar with [Fou10] will notice that the arguments that go into the proofs of Proposition 4.20 and Corollary 4.23 are similar to those of [Fou10, §5], and that the hypothesis **H.stz** is not assumed in loc.cit. The point of this remark is to explain why **H.stz** is indeed necessary also in the proof of [Fou10, Lemma 5.14] (in fact implied by the arguments therein), more precisely to indicate how the norm-compatibility of $\{\kappa_{[n,a]}\}$ as a varies cannot be used alone in order to conclude with Corollary 4.23 without assuming **H.stz**.

For every positive integer $a >> 0$, fix a place w_a of K_a above v, such that $w_{a'} \mid w_a$ for $a \geq a'$. Let ϕ_a be the natural map

$$\phi_a:\ H^1((K_a)_{w_a}, \mathrm{F}_v^+(T_{k,m})) \longrightarrow H^1((K_a)_{w_a}, T_{k,m}).$$

Set $T_k = \mathbb{T}/x^k\mathbb{T}$ and define r_a, δ_a

$$H^1((K_a)_{w_a}, \mathrm{F}_v^+(T_k)) \xrightarrow{r_a} H^1((K_a)_{w_a}, \mathrm{F}_v^+(T_{k,m})) \xrightarrow{\delta_a} H^2((K_a)_{w_a}, \mathrm{F}_v^+(T_k))$$

as the natural homomorphisms in the $G_{(K_a)_{w_a}}$-cohomology of the short exact sequence

$$0 \longrightarrow \mathrm{F}_v^+(T_k) \xrightarrow{\pi^m} \mathrm{F}_v^+(T_k) \longrightarrow \mathrm{F}_v^+(T_{k,m}) \longrightarrow 0.$$

By Remark 4.21, there exists $\mathfrak{x}_a \in H^1((K_a)_{w_a}, \mathrm{F}_v^+(T_{k,m}))$ such that $\phi_a(\mathfrak{x}_a) = \mathrm{loc}_{w_a}(\kappa_{[n,a]})$. Note that ϕ_a is injective only if **H.stz** holds true and hence $\mathfrak{x}_a$

is not uniquely determined. In particular, the collection $\{\mathfrak{x}_a\}$ is not necessarily norm-coherent as a varies. This is first of the problems. In order to check that

$$\begin{aligned}\mathrm{loc}_{w_a}(\kappa_{[n,a]}) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}((K_a)_{w_a}, T_{k,m}) &= \mathrm{im}\left(H^1((K_a)_{w_a}, \mathrm{F}_v^+(\mathbb{T})) \to H^1((K_a)_{w_a}, T_{k,m})\right)\\ &\subset \mathrm{im}\left(H^1((K_a)_{w_a}, \mathrm{F}_v^+(T_k)) \to H^1((K_a)_{w_a}, T_{k,m})\right)\end{aligned}$$

one attempts to choose $\mathfrak{x}_a$ in a way that $\delta_a(\mathfrak{x}_a) = 0$ as follows: Argue (using the fact that the module $H^2((K_a)_{w_a}, \mathrm{F}_v^+(T_k))$ is of finite order bounded independently of a, when x is *unexceptional* in an appropriate sense) that there is a $\mathfrak{b} >> 0$ such that

$$\mathrm{im}\left(H^2((K_\mathfrak{b})_{w_\mathfrak{b}}, \mathrm{F}_v^+(T_k)) \xrightarrow{\mathrm{cor}} H^2((K_a)_{w_a}, \mathrm{F}_v^+(T_k))\right) = 0, \tag{4.4}$$

so that for $\mathfrak{x}_a$ chosen as $\mathfrak{x}_a = \mathrm{cor}(\mathfrak{x}_\mathfrak{b})$ for a $\mathfrak{b}$ satisfying (4.4), we would conclude that

$$\delta_a(\mathfrak{x}_a) := \mathrm{cor}(\delta_\mathfrak{b}(\mathfrak{x}_\mathfrak{b})) = 0.$$

However, if choosing $\mathfrak{b}$ as in (4.4) was possible, then it would follow that

$$H^2((K_a)_{w_a}, \mathrm{F}_v^+(T_k)) = 0, \tag{4.5}$$

since we have a commutative diagram

$$\begin{tikzcd}
H^2((K_a)_{w_a}, \mathrm{F}_v^+(T_k)\otimes\Lambda^{\mathrm{ac}}) \arrow[rr, two heads] \arrow[dr] & & H^2((K_a)_{w_a}, \mathrm{F}_v^+(T_k)) \\
& H^2((K_\mathfrak{b})_{w_\mathfrak{b}}, \mathrm{F}_v^+(T_k)) \arrow[ur, "\mathrm{cor}"'] &
\end{tikzcd}$$

where the surjection is because the cohomological dimension of $G_{(K_a)_{w_a}}$ is 2. Now by local duality, it is easy to see that (4.5) is equivalent to asking **H.stz**.

4.3.3. *Local properties at primes dividing* N. Throughout this section Assumption 3.4 is in effect; see Remark 3.5 for the content of this assumption. Suppose $n \in \mathcal{N}_{k,m,r}$. Let $v \mid N$ be a place of K and w be any place of K_α above v.

Proposition 4.25.

(i) $\mathrm{loc}_w(\kappa_{[n,\alpha]}) \in \ker(H^1((K_\alpha)_w, T_{k,m}) \longrightarrow H^1((K_\alpha)_w^{\mathrm{ur}}, T_{k,m}))$,
(ii) $\mathrm{loc}_v(\kappa_n) \in \ker(H^1(K_v, T_{k,m,r}) \longrightarrow H^1(K_v^{\mathrm{ur}}, T_{k,m,r}))$.

Proof. The proof of Proposition 4.18 goes through verbatim, except in the final paragraph one needs to replace $T_{k,m}$ (resp., $T_{k,m,r}$) by $T_{k,m}^{I_v}$ (resp., $T_{k,m,r}^{I_v}$), when they appear in the cohomology computed for the group $\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v)$. $\square$

Proposition 4.26. $\mathrm{loc}_v(\kappa_n) \in H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_v, T_{k,m,r})$. *(That is to say,* $\mathrm{loc}_v(\kappa_n)$ *is in the image of* $H^1_{\mathrm{ur}}(K_v, \mathbb{T} \otimes \Lambda^{\mathrm{ac}})$ *under the natural map induced from* $\mathbb{T} \otimes \Lambda^{\mathrm{ac}} \twoheadrightarrow T_{k,m,r}$.

Proof. The commutative diagram with exact rows

$$\begin{tikzcd} H^1_{\mathrm{ur}}(K_v, \mathbb{T} \otimes \Lambda^{\mathrm{ac}}) \arrow[r] & H^1(K_v, \mathbb{T} \otimes \Lambda^{\mathrm{ac}}) \arrow[r] \arrow[d] & H^1(K_v^{\mathrm{ur}}, \mathbb{T} \otimes \Lambda^{\mathrm{ac}}) \arrow[d] \\ H^1_{\mathrm{ur}}(K_v, T_{k,m,n}) \arrow[r] & H^1(K_v, T_{k,m,n}) \arrow[r] & H^1(K_v^{\mathrm{ur}}, T_{k,m,n}) \end{tikzcd}$$

shows that the vertical arrow on the left induces a map

$$H^1_{\mathrm{ur}}(K_v, \mathbb{T} \otimes \Lambda^{\mathrm{ac}}) \longrightarrow H^1_{\mathrm{ur}}(K_v, T_{k,m,r}). \tag{4.6}$$

To conclude the proof, it suffices to prove that this map is surjective. On the other hand, the map (4.6) is

$$H^1(K_v^{\mathrm{ur}}/K_v, (\mathbb{T} \otimes \Lambda^{\mathrm{ac}})^{I_v}) \longrightarrow H^1(K_v^{\mathrm{ur}}/K_v, T_{k,m,r}^{I_v}),$$

and this map is surjective thanks to Lemma 4.27 below and the facts that

- I_v acts trivially on Λ^{ac},
- the cohomological dimension of $\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v)$ is one.

□

Lemma 4.27. *The natural map* $\mathbb{T}^{I_v} \to T_{k,m}^{I_v}$ *is surjective.*

Proof. Under our running assumptions, $\mathbb{T}$ fits in an exact sequence of $\mathcal{R}[[G_v]]$-modules

$$0 \longrightarrow \mathcal{R}(1) \otimes \mu \longrightarrow \mathbb{T} \longrightarrow \mathcal{R} \longrightarrow 0,$$

where $\mu^2 = 1$ and μ is unramified.

In the case $\mu \neq \mathrm{id}$, the proof of Proposition 3.2 shows that

$$\mathbb{T} = (\mathcal{R}(1) \otimes \mu) \oplus \mathcal{R}$$

as G_v-modules and thus $\mathbb{T}^{I_v} = \mathbb{T}$ and $\mathbb{T}^{I_v} = \mathbb{T} \twoheadrightarrow T_{k,m}$ as desired.

In the case $\mu = \mathrm{id}$, the I_v-cohomology of the short exact sequence

$$0 \longrightarrow R(1) \longrightarrow T \longrightarrow R \longrightarrow 0$$

(for $R = \mathcal{R}, R_{k,m}$ and $T = \mathbb{T}, T_{k,m}$) yields the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{R}(1) \arrow[r, "\alpha_0"] \arrow[d] & \mathbb{T}^{I_v} \arrow[r, "\beta"] \arrow[d] & \mathcal{R} \arrow[r, "\partial"] \arrow[d] & H^1(I_v, \mathcal{R}(1)) \arrow[r, "\alpha"] \arrow[d] & H^1(I_v, \mathbb{T}) \arrow[d] \\ 0 \arrow[r] & R_{k,m}(1) \arrow[r, "\bar{\alpha}_0"'] & T_{k,m}^{I_v} \arrow[r, "\bar{\beta}"'] & R_{k,m} \arrow[r, "\bar{\partial}"'] & H^1(I_v, R_{k,m}(1)) \arrow[r, "\bar{\alpha}"'] & H^1(I_v, T_{k,m}) \end{tikzcd}$$

Under the running assumptions, the proof of Lemma 3.11 shows that the map ∂ is surjective, in particular non-zero and therefore injective. This shows the map β is the zero map, hence α_0 is surjective and therefore an isomorphism. The proof of Proposition 3.9 shows that $\bar{\alpha}$ is injective and it follows as above that the map $\bar{\alpha}_0$ is an isomorphism. Proof now follows by the surjectivity of the left-most vertical arrow. □

Theorem 4.28. *There is a Kolyvagin system* $\tilde{\boldsymbol{\kappa}} \in \overline{\mathbf{KS}}(\mathbb{T} \otimes \Lambda^{\mathrm{ac}}, \mathcal{F}_{\mathrm{Gr}})$ *such that*

$$\tilde{\kappa}_1 = \kappa_1 = \{\mathfrak{z}_s\} \in \varprojlim_s H^1(K_s, \mathbb{T}).$$

Proof. Recall that

$$\overline{\mathbf{KS}}(\mathbb{T} \otimes \Lambda^{\mathrm{ac}}, \mathcal{F}_{\mathrm{Gr}}) = \varprojlim \mathbf{KS}(T_{k,m,r}, \mathcal{F}_{\mathrm{Gr}}, \mathcal{P}_{k,m,r}).$$

Denote by $\kappa_n^{(k,m,r)}$ what we have called κ_n in Definition 4.17. We have verified above that $\kappa_n^{(k,m,r)} \in H^1_{\mathcal{F}_{\mathrm{Gr}}(n)}(K, T_{k,m,r})$. To finish off the proof one compares, as carried out in [Nek92, §7], the images of the classes $\mathrm{loc}_\ell\left(\kappa_n^{(k,m,r)}\right)$ and $\mathrm{loc}_\ell\left(\kappa_{n\ell}^{(k,m,r)}\right)$ under the identifications

$$H^1_f(K_\lambda, T_{k,m,r}) \xrightarrow{\sim} T_{k,m,r} \xleftarrow{\sim} H^1_s(K_\lambda, T_{k,m,r}), \tag{4.7}$$

and modifies $\kappa_n^{(k,m,r)}$ slightly as in [How04, Theorem 1.7.5] so as to obtain

$$\tilde{\kappa}_n^{(k,m,r)} \in H^1_{\mathcal{F}_{\mathrm{Gr}}(n)}(K, T_{k,m,r}) \otimes \mathcal{G}(n)$$

satisfying the desired condition

$$\phi_\lambda^{\mathrm{fs}}\left(\left(\mathrm{loc}_\ell(\tilde{\kappa}_n^{(k,m,r)}\right)\right) = \mathrm{loc}_\ell\left(\tilde{\kappa}_{n\ell}^{(k,m,r)}\right).$$

□

5. Howard's main conjecture

Let R_∞ be the ring $\mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda^{\mathrm{ac}}$. In this section, we record the standard application of the big Heegner point Kolyvagin system $\tilde{\boldsymbol{\kappa}}$ that we have constructed in §4.3. We omit the proofs as they follow closely the proofs in [Fou10, §6.3], except that we do not have the unwanted factor $\alpha \in \mathcal{R}$ that appear in the statements[2,3] of [Fou10, Theorem B(iii), Theorem 3].

[2]The attentive reader will notice that the extra factor α is not explicit in the statement of [Fou10, Theorem B(iii)], however that Fouquet's element z_∞ differ from the $\mathfrak{z}_\infty$ defined below by a factor of α.

[3]Note that the extra factor α in Fouquet's[Fou10] arguments is needed to obtain Kolyvagin systems for each specialization of $\mathbb{T}$. Once one obtains those (in our case, they descend from our big Heegner point Kolyvagin system), the arguments of §6 in loc.cit. carry out verbatim.

For a finite extension L of K, let $\tilde{H}^i_f(L,\mathbb{T})$ be Nekovář's extended Selmer group defined in [Nek06, §6] and let

$$\tilde{H}^i_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}) = \varprojlim_s \tilde{H}^i_f(K_s,\mathbb{T}).$$

It follows by [Nek06, Lemma 9.6.3] and [How07, Lemma 2.4.4] that $\tilde{H}^i_f(K_s,\mathbb{T}) = H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_s,\mathbb{T})$ and hence we may define an element

$$\mathfrak{z}_\infty = \{\mathfrak{z}_s\} \in \tilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}).$$

Assume henceforth that the ring R_∞ is a regular ring, so that $(R_\infty)_\mathfrak{p}$ is a DVR for every hight one prime $\mathfrak{p}$ of R_∞. If M is a finitely generated torsion R_∞-module, we may then define the characteristic ideal

$$\mathrm{char}(M) = \prod_\mathfrak{p} \mathfrak{p}^{\mathrm{length}(M_\mathfrak{p})}.$$

For a general R_∞-module M, let M_{tors} denote the R_∞-torsion submodule.

Theorem 5.1.

$$\mathrm{char}\left(\tilde{H}^2_{f,\mathrm{Iw}}(K_\infty,\mathbb{T})_{\mathrm{tors}}\right) \mid \mathrm{char}\left(\tilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T})/R_\infty\mathfrak{z}_\infty\right)^2.$$

It is reasonable to expect that one could adapt the arguments of [Arn11] in order to prove a similar statement assuming only that the ring $\mathcal{R}$ is normal.

Acknowledgements. The author wishes to thank Olivier Fouquet, Barry Mazur and Jan Nekovář, discussions with whom led the author to this work. The author acknowledges a Marie Curie Grant EU-FP7 230668 and a TÜBİTAK Grant 109T662.

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