

Deformations of Kolyvagin systems

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ABSTRACT. Mazur and Rubin prove the existence of Kolyvagin systems for a general class of mod p Galois representations $\bar{\rho}$. Furthermore, they also prove (under certain hypotheses) that these Kolyvagin systems may be deformed to Kolyvagin systems for a deformation of $\bar{\rho}$ to a discrete valuation ring. The goal of this article is to achieve this for larger coefficient rings. More specifically, we carry this out for two types of deformations of Galois representations: the universal deformation when the deformation problem is unobstructed, and deformations to a two-dimensional Gorenstein ring. This generalizes the works of Howard on Heegner points and Ochiai on Kato's Euler system. We give explicit arithmetic applications of our result on the existence of 'big' Kolyvagin systems, such as the interpolation of Kato's Euler system in families (not necessarily p -ordinary) of modular Galois representations.

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2000 *Mathematics Subject Classification.* 11G05; 11G10; 11G40; 11R23; 14G10.

Key words and phrases. Euler systems, Kolyvagin systems, Deformations of Kolyvagin systems, Equivariant Tamagawa Number Conjecture.

1. INTRODUCTION

Since it was introduced by Kolyvagin [Kol90], the Euler system machinery has been used by many to obtain important results in arithmetic. Kolyvagin used the Heegner point Euler system in [Kol91a, Kol91b] to bound the order and determine the structure of the Tate-Shafarevich groups of elliptic curves, making an important progress towards the Birch and Swinnerton-Dyer conjecture. Rubin generalized Kolyvagin's machinery to apply it in the realm of Iwasawa theory. He used in [Rub91] the elliptic unit Euler system to prove the main conjectures of Iwasawa theory for CM elliptic curves, and in [Lan90] to give an elementary proof of the classical main conjectures (originally proved by Mazur and Wiles [MW84]) using the cyclotomic unit Euler system. Later Kato [Kat04] constructed an Euler system for elliptic modular forms using the Beilinson elements in the K_2 of modular curves, and deduced one of the divisibilities of the main conjectures of Iwasawa theory in this setting.

Iwasawa theory of \mathbb{Z}_p -extensions may be seen as a particular instance of Mazur's general theory of Galois deformations. Greenberg [Gre94] takes this perspective to generalize Iwasawa's theory to a more general study of *deformations of motives*. Before we relate the content of this article to Greenberg's theory, we provide a quick overview of Mazur's theory of Galois deformations; see [Maz89, dSL97, Gou01] for details.

Fix forever an odd prime p . Let Φ be a finite extension of \mathbb{Q}_p and \mathcal{O} be the ring of integers of Φ . Let $\varpi \in \mathcal{O}$ be a uniformizer, and let $k = \mathcal{O}/\varpi$ be its residue field. Consider the following category \mathbf{C} :

- An object of \mathbf{C} is a commutative, complete, local, Noetherian \mathcal{O} -algebra A whose residue field $k_A = A/\mathfrak{m}_A$ is isomorphic to k , where \mathfrak{m}_A denotes the maximal ideal of A .
- A morphism $f : A \rightarrow B$ in \mathbf{C} is a local \mathcal{O} -algebra morphism.

Let Σ be a finite set of places of \mathbb{Q} that contains p and ∞ . Let $G_{\mathbb{Q},\Sigma}$ denote the Galois group of the maximal extension \mathbb{Q}_Σ of \mathbb{Q} unramified outside Σ . Fix an absolutely irreducible, continuous Galois representation

$$\bar{\rho} : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_n(k),$$

and let \bar{T} be the representation space (so that \bar{T} is an n -dimensional k -vector space on which $G_{\mathbb{Q},\Sigma}$ acts continuously).

Let $F_{\bar{\rho}} : \mathbf{C} \longrightarrow \mathbf{Sets}$ be the functor defined as follows. For every object A of \mathbf{C} , $F_{\bar{\rho}}(A)$ is the set of continuous homomorphisms

$$\rho_A : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_n(A)$$

that satisfy $\rho_A \otimes_A k \cong \bar{\rho}$, taken modulo conjugation by the elements of $\mathrm{GL}_n(A)$. For every morphism $f : A \rightarrow B$ in \mathbf{C} , $F_{\bar{\rho}}(f)(\rho_A)$ is the $\mathrm{GL}_n(B)$ -conjugacy class of $\rho_A \otimes_A B$.

Theorem (Mazur). *The functor $F_{\bar{\rho}}$ is representable.*

In other words, there is a ring $R(\bar{\rho}) \in \mathrm{Ob}(\mathbf{C})$ and a continuous representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(R(\bar{\rho}))$$

such that for every $A \in \mathrm{Ob}(\mathbf{C})$ and any continuous representation $\rho_A : G_{\mathbb{Q},\Sigma} \rightarrow \mathrm{GL}_n(A)$, there is a unique morphism $f_A : R(\bar{\rho}) \rightarrow A$ such that

$$\rho \otimes_{R(\bar{\rho})} A \cong \rho_A.$$

The ring $R(\bar{\rho})$ is called the universal deformation ring and ρ the universal deformation of $\bar{\rho}$.

Let $\text{Ad}(\bar{\rho})$ be the adjoint representation. Consider the following hypothesis:

$$(\mathbf{H.nOb}) \ H^2(G_{\mathbb{Q},\Sigma}, \text{Ad}(\bar{\rho})) = 0.$$

We say that the deformation problem for $\bar{\rho}$ is *unobstructed* if $(\mathbf{H.nOb})$ holds true. In this case Mazur shows that

$$R(\bar{\rho}) \cong \mathcal{O}[[X_1, \dots, X_d]]$$

where $d = \dim_{\mathbb{k}}(H^1(G_{\mathbb{Q},\Sigma}, \text{Ad}(\bar{\rho})))$.

Example 1.1. Suppose E/\mathbb{Q} is an elliptic curve. Let Σ be the set of primes that consists of primes at which E has bad reduction, p and ∞ . Let $\bar{T} = E[p]$, the p -torsion of E and

$$\bar{\rho}_E : G_{\mathbb{Q},\Sigma} \longrightarrow \text{GL}_2(\mathbb{F}_p)$$

the associated Galois representation. Flach [Fla92] shows that the deformation problem for $\bar{\rho}$ is unobstructed and $R(\bar{\rho}) \cong \mathbb{Z}_p[[X_1, X_2, X_3]]$ if the following holds:

- $\bar{\rho}_E$ is surjective,
- $H^0(\mathbb{Q}_\ell, \bar{T} \otimes \bar{T}) = 0$ for all $\ell \in \Sigma$,
- p does not divide $\Omega^{-1}L(\text{Sym}^2(E), 2)$, where $\Omega = \Omega(\text{Sym}^2(E), 2)$ is the transcendental period.

Example 1.2. Let f be an elliptic newform of level N , weight $k > 2$ and character ψ . Let K be the number field generated by the Fourier coefficients of f and \mathcal{O}_K be its ring of integers. For a prime \wp of K above p , let $\mathbb{k} = \mathcal{O}_K/\wp$ and $\mathcal{O} = W(\mathbb{k})$, the Witt vectors of \mathbb{k} . Let

$$\bar{\rho} = \bar{\rho}_{f,\wp} : G_{\mathbb{Q},\Sigma} \longrightarrow \text{GL}_2(\mathbb{k})$$

be the Galois representation attached to f by Deligne. Then Weston [Wes04] shows for almost all choices of a prime \wp of K , the deformation problem for $\bar{\rho}$ is unobstructed and $R(\bar{\rho}) \cong \mathcal{O}[[X_1, X_2, X_3]]$.

One may also study a subclass of deformations of a given $\bar{\rho}$, rather than the full deformation space $R(\bar{\rho})$. The following paragraph illustrates a particular case which has been much studied by many authors. Suppose

$$\bar{\rho} : G_{\mathbb{Q},\Sigma} \longrightarrow \text{GL}_2(\mathbb{k})$$

is p -ordinary and a p -distinguished, in the sense that the restriction of $\bar{\rho}$ to a decomposition group at p is reducible and non-scalar. Assume further that $\bar{\rho}$ is odd, i.e., $\det(\bar{\rho})(c) = -1$ where c is any complex conjugation. Then Serre's conjecture [Ser87] (as proved in [KW09, Kis09a]) implies that $\bar{\rho}$ arises from an ordinary newform f as in Example 1.2. Hida associates in [Hid86b, Hid86a] such f a family of ordinary modular forms and a Galois representation \mathcal{T} attached to the family, with coefficients in the *universal ordinary Hecke algebra* \mathfrak{H} . Thanks to the “ $R = T$ ” theorems proved in [Wil95, TW95] (and their refinements) it follows that \mathfrak{H} is the universal ordinary deformation ring of $\bar{\rho}$ parametrizing all ordinary deformations of $\bar{\rho}$. Ochiai [Och05] (resp., Howard [How07]) has studied the Iwasawa theory of this family of Galois representations by interpolating Kato's Euler system (resp., Heegner points) for each member of the family to a ‘big’ Euler system for the whole family.

The main goal of the current article is to generalize the work of Ochiai [Och05] and Howard [How07] to more general Galois representations and more general class of deformation rings.

The general Euler system machinery takes an Euler system (which is a collection of cohomology classes that satisfies certain norm-compatibility conditions) as an input and produces

what Kolyvagin calls the *derivative classes*. In [MR04], Mazur and Rubin slightly modify the derivative classes so as to obtain what they call *Kolyvagin systems*. Using Kolyvagin systems one obtains the bounds on Selmer groups that we seek for. As powerful the Euler system machinery is, as difficult it is to construct an Euler system for a general Galois representation, although some folklore conjectures (such as the Bloch-Kato conjectures, c.f., [BK90, FPR94]) hint at the existence of Euler systems in great generality. On the other hand, Mazur and Rubin prove in [MR04, Theorem 5.1.1] that Kolyvagin systems do exist for a very general class of $\text{mod } p$ Galois representations. Furthermore, they prove a similar result for deformations of these Galois representations to discrete valuation rings. Later in [Büy11b], the author extended their result by proving that these Kolyvagin systems in fact may be deformed to the cyclotomic Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$, where Γ is the Galois group of the cyclotomic \mathbb{Z}_p -extension $\mathbb{Q}_\infty/\mathbb{Q}$.

In this paper we will study the deformation problem for Kolyvagin systems to one of the following choices of rings:

- (i) $\mathfrak{R} = \mathcal{R}[[\Gamma]]$, where \mathcal{R} is a dimension-2 Gorenstein \mathcal{O} -algebra with a regular sequence $\{\varpi, X\}$ such that \mathcal{R}/X is a finitely generated torsion-free \mathcal{O} -module.
- (ii) $R = \mathcal{O}[[X_1, X_2, X_3]]$ (which we will think of as an unobstructed universal deformation ring of a two dimensional $\text{mod } \varpi$ Galois representation $\bar{\rho}$ in examples.)

See Theorem A below for our main result in this direction.

Before we describe our results we fix some notation. Let \mathfrak{m} (resp., \mathcal{M}) be the maximal ideal of \mathfrak{R} (resp., of R) and $k = \mathfrak{R}/\mathfrak{m}$ (resp., $k = R/\mathcal{M}$) be the residue field. When the coefficient ring we are interested in is the ring \mathfrak{R} as in (i) above, we let \mathcal{T} be a free \mathcal{R} -module of finite rank which is endowed with a continuous $G_\mathbb{Q}$ -action, unramified outside a finite set of primes. Set $\mathfrak{T} = \mathcal{T} \otimes_{\mathbb{Z}_p} \Lambda$, where we allow $G_\mathbb{Q}$ act on both factors. When the coefficient ring we are interested in is R as in (ii), we let \mathbb{T} be a free R -module of finite rank endowed with a continuous $G_\mathbb{Q}$ -action unramified outside a finite number of primes. In either case, we let $\bar{T} = \mathfrak{T}/\mathfrak{m}$ (resp., $\bar{T} = \mathbb{T}/\mathcal{M}$) and define $\chi(\mathfrak{T}) = \dim_k \bar{T}^-$ (resp., $\chi(\mathbb{T})$), where \bar{T}^- is the (-1) -eigensubspace of \bar{T} under the action of a fixed complex conjugation.

The following hypotheses will play a role in what follows:

(H1) \bar{T} is an absolutely irreducible $G_\mathbb{Q}$ -module.

(H2) There is a $\tau \in G_\mathbb{Q}$ such that τ acts trivially on μ_{p^∞} and the R -module $\mathbb{T}/(\tau - 1)\mathbb{T}$ (resp., the \mathfrak{R} -module $\mathfrak{T}/(\tau - 1)\mathfrak{T}$) is free of rank one.

(H3) $H^0(\mathbb{Q}, \bar{T}) = H^0(\mathbb{Q}, \bar{T}^*) = 0$, where $\bar{T}^* = \text{Hom}(\bar{T}, \mu_p)$.

(H4) Either

- (i) $\text{Hom}_{\mathbb{F}_p[[G_\mathbb{Q}]]}(\bar{T}, \bar{T}^*) = 0$, or
- (ii) $p > 4$.

(H.Tam) For all bad primes ℓ ,

- (i) $H^0(\mathbb{Q}_\ell, \bar{T}) = 0$.
- (ii) $H^0(I_\ell, A)$ is p -divisible.

(H.nA) $H^0(\mathbb{Q}_p, \bar{T}^*) = 0$.

Remark 1.3. The hypotheses **(H1)**-**(H4)** are also present in [MR04]. **(H.Tam)** is used to check that the unramified local conditions are cartesian (in a sense made precise below), most importantly in the proof that the map β that appears in Lemma 4.6 is injective. Although one may possibly verify this fact under less restrictive hypothesis, the assumption **(H.Tam)** is

simple to state, easy to check and thus allows one to produce many interesting examples where Theorem A applies; c.f., Proposition 5.1 below.

Remark 1.4. When \mathcal{T} is the self-dual Galois representation attached to a (twisted) Hida family with coefficients in the universal ordinary Hecke algebra \mathcal{R} (as studied in [How07]), the hypothesis **(H.Tam)**(ii) asks that there is a single member f of the twisted Hida family such that the Tamagawa number (as defined by [Fontaine-PR]) $c_\ell(f)$ is prime to p . As explained in [Büy11a, §3], this in turn implies that the Tamagawa number $c_\ell(g)$ is prime to p for every member g of the twisted family.

See §5.1 for a discussion of the content of the hypotheses **(H.Tam)** and **(H.nA)** when \bar{T} is the mod p Galois representation attached to an elliptic curve E/\mathbb{Q} .

For \mathbb{T} (resp., \mathfrak{T}) as above, let $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ (resp., $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$) be the R -module (resp., the \mathfrak{R} -module) of *big* Kolyvagin systems for \mathbb{T} (resp., for \mathfrak{T}) and the *canonical Selmer structure*; see §2 and §3.2 for precise definitions of these objects.

Theorem A (See Theorem 3.12 below). *Suppose $\chi(\mathfrak{T}) = \chi(\mathbb{T}) = 1$. Under the hypotheses **(H1-H4)**, **(H.Tam)** and **(H.nA)**,*

- (i) *the R -module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one, generated by a Kolyvagin system κ whose image $\bar{\kappa} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero,*
- (ii) *the \mathfrak{R} -module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one. When the ring \mathcal{R} is regular, the module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is generated by κ whose image $\bar{\kappa} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero.*

In Theorem A, \mathbb{T} (resp., \mathfrak{T}) should be thought of as a family of Galois representations and the conclusion of Theorem A as an assertion that the Kolyvagin systems for each individual member of the family \mathbb{T} (resp., \mathfrak{T}) interpolate to give rise to a ‘big’ Kolyvagin system.

We remark that the arguments used in the proof of Theorem A generalize without any effort to handle a general regular ring (not necessarily of relative dimension 3 over \mathcal{O} , as R above is). However, any significant arithmetic application of our general theorem is restricted to the case when the big Kolyvagin system for \mathbb{T} (resp., for \mathfrak{T}) that is proved to exist in Theorem A interpolates Kolyvagin systems which are explicitly related to L -values. At the moment this is only possible when the residual representation \bar{T} is two dimensional and $\chi(\bar{T}) = 1$, in which case the unobstructed universal deformation ring is R . Note that when \bar{T} is two dimensional and $\chi(\bar{T}) = 1$, we know that \bar{T} is modular and the recent advances in modularity lifting theorems and progress towards the Fontaine-Mazur conjecture (due to Emerton [Eme11] and Kisin [Kis09b]) show that the big Kolyvagin system for the universal deformation \mathbb{T} of \bar{T} indeed interpolates Kato’s Kolyvagin systems for elliptic modular forms whose associated Galois representations are congruent to \bar{T} .

To be more precise on the last point we made, let E/\mathbb{Q} be an elliptic curve, $\bar{T} = E[p]$ be the p -torsion subgroup of $E(\bar{\mathbb{Q}})$ and $\bar{\rho} = \bar{\rho}_E$ the mod p Galois representation on \bar{T} . Define also $R = R(\bar{\rho})$ to be the universal deformation ring of $\bar{\rho}$. Suppose that E satisfies **(H.nOb)** so that $R \cong \mathbb{Z}_p[[T_1, T_2, T_3]]$. The universal deformation representation \mathbb{T} is a free R -module of rank two. Until the end of this introduction, \mathbb{T} will stand for this particular Galois representation; see the main body of the text for the most general form of the results we record in this section. Suppose also that \mathbb{T} satisfies the hypotheses **(H1)-(H4)** as well as **(H.Tam)**(i) and **(H.nA)**; see §5.1 for the content of these assumptions in this particular setting.

Let $f = \sum a_n q^n$ be a newform of weight $\omega \geq 2$ and let \mathcal{O}_f be the finite flat normal extension of \mathbb{Z}_p which the a_n 's generate. Let

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}_f)$$

be the Galois representation attached to E by Deligne and T_f be the free \mathcal{O}_f -module of rank two on which $G_{\mathbb{Q}}$ acts via ρ_f . Suppose that $\bar{\rho}_f \cong \bar{\rho}$, so that by the universality of R there is a ring homomorphism $\varphi_f : R \rightarrow \mathcal{O}_f$ which induces an isomorphism $\mathbb{T} \otimes_{\varphi_f} \mathcal{O}_f \cong T_f$ and a map

$$\overline{\mathrm{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \xrightarrow{\varphi_f} \overline{\mathrm{KS}}(T_f, \mathcal{F}_{\mathrm{can}}, \mathcal{P}).$$

For each f as above, Kato in [Kat04] constructed a Kolyvagin system $\kappa^{\mathrm{Kato},(f)} \in \overline{\mathrm{KS}}(T_f, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ using Beilinson's elements. As a consequence of Theorem A above, we prove that a generator κ of the R -module $\overline{\mathrm{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ interpolates $\kappa^{\mathrm{Kato},(f)}$ as f varies, generalizing the results of [Och05, How07] where similar results were obtained for the universal ordinary deformation.

Theorem B (Theorem 5.2).

- (i) $\kappa^{\mathrm{Kato},(f)} = \lambda_f \cdot \varphi_f(\kappa)$ for some $\lambda_f \in \mathcal{O}_f$.
- (ii) Assume further that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible and non-scalar. Then the leading term κ_1 of κ is non-zero.

The following result is the standard application of the Kolyvagin system machinery, proved essentially in [Och05]. Let $\mathcal{F}_{\mathrm{can}}^*$ be the dual Selmer structure on \mathbb{T}^* , see Definition 2.4 for a definition of the dual Selmer structure. For any abelian group N , let N^{\vee} denote the Pontryagin dual. If M is a finitely generated torsion R -module, set

$$\mathrm{char}(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{length}(M_{\mathfrak{p}})}$$

where the product is over height one primes of R .

Theorem C. *The cofinitely generated R -module $H_{\mathcal{F}_{\mathrm{can}}^*}^1(\mathbb{Q}, \mathbb{T}^*)$ is cotorsion and*

$$\mathrm{char}\left(H_{\mathcal{F}_{\mathrm{can}}^*}^1(\mathbb{Q}, \mathbb{T}^*)^{\vee}\right) \mid \mathrm{char}\left(H_{\mathcal{F}_{\mathrm{can}}}^1(\mathbb{Q}, \mathbb{T})/R \cdot \kappa_1\right),$$

where κ_1 is the leading term of a generator κ of the R -module $\overline{\mathrm{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ of big Kolyvagin systems.

This statement is closely related to Greenberg's main conjectures [Gre94] for deformations of motives. To explain, we give the following definition and propose the conjecture that follows in order to define the “correct Selmer structure” $\mathcal{F}_{\mathrm{BK}}$ on \mathbb{T} as above:

Definition 1.5. Let \mathfrak{D} be a finite, flat, normal extension of \mathbb{Z}_p . A *pseudo-geometric specialization* is a ring homomorphism $\varphi : R \rightarrow \mathfrak{D}$ such that the $G_{\mathbb{Q}_p}$ -representation is $\mathbb{T} \otimes_{\varphi} \mathrm{Frac}(\mathfrak{D})$ is de Rham.

Conjecture D. *There is an R -submodule (the Bloch-Kato submodule)*

$$H_{\mathcal{F}_{\mathrm{BK}}}^1(\mathbb{Q}_p, \mathbb{T}) \subset H^1(\mathbb{Q}_p, \mathbb{T})$$

with the following properties:

- (i) (Interpolation) Let $\varphi : R \rightarrow \mathfrak{O}$ be a pseudo-geometric specialization into a finite, flat, normal extension \mathfrak{O} of \mathbb{Z}_p and $T = \mathbb{T} \otimes_{\varphi} \mathfrak{O}$, $V = T \otimes \mathbb{Q}_p$. The Bloch-Kato submodule $H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}_p, \mathbb{T})$ maps into the Bloch-Kato subgroup

$$H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}_p, T) := \ker (H^1(\mathbb{Q}_p, T) \rightarrow H^1(\mathbb{Q}_p, V \otimes B_{\text{cris}}))$$

with finite cokernel,

- (ii) (Pančičkin condition) $H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}_p, \mathbb{T})$ is a free R -module of rank 1.
- (iii) $H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}, \mathbb{T}) := \ker \left(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathbb{T}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathbb{T})}{H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}_p, \mathbb{T})} \right) = 0$.

See [Pot11] for progress related to this conjecture.

Set $\text{loc}_p^s : H^1(\mathbb{Q}, \mathbb{T}) \rightarrow H_s^1(\mathbb{Q}_p, \mathbb{T}) := H^1(\mathbb{Q}_p, \mathbb{T})/H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}_p, \mathbb{T})$. Using the property (iii) in Conjecture D, we conclude that:

Corollary E (Corollary 5.15). $\text{char} \left(H_{\mathcal{F}_{\text{BK}}}^1(\mathbb{Q}, \mathbb{T}^*)^{\vee} \right) \mid \text{char} \left(H_s^1(\mathbb{Q}_p, \mathbb{T})/R \cdot \text{loc}_p^s(\kappa_1) \right)$.

We call the ideal $\mathcal{L}(\kappa) := \text{char} \left(H_s^1(\mathbb{Q}_p, \mathbb{T})/R \cdot \text{loc}_p^s(\kappa_1) \right)$ the *Kolyvagin-constructed p -adic L -function*. This should be thought of as a generalization of Perrin-Riou's [PR95] module of algebraic p -adic L -function, whose definition she gives for the cyclotomic deformation of a motive. This choice of terminology is justified thanks to the following *interpolation property*:

Theorem F (Proposition 5.17, Theorem 5.18). *Suppose that the R -module $H_s^1(\mathbb{Q}_p, \mathbb{T})$ is torsion-free. Suppose also that the elliptic curve E has good ordinary reduction at p .*

- (i) $\mathcal{L}(\kappa) \neq 0$.
- (ii) Assume further that the R -module $H_s^1(\mathbb{Q}_p, \mathbb{T})$ is free. Then for every elliptic modular form f as in Theorem B,

$$(1.1) \quad [\mathcal{O}_f : \varphi_f(\mathcal{L}(\kappa))] = \lambda_f^{-1} \cdot \# \left(\frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s(\kappa_1^{\text{Kato}, (f)})} \right).$$

Kato has related the values $\# \left(\frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s(\kappa_1^{\text{Kato}, (f)})} \right)$ which appear in the statement of

Theorem F to the value of the L -function attached to f at the central critical point. The statement of Theorem 5.18 therefore suggests that these values should interpolate, as the classes $\lambda_f^{-1} \kappa_1^{\text{Kato}, (f)}$ do interpolate to κ_1 . This hints at the existence of a very general p -adic L -function. Furthermore, we note that the points $\varphi_f : \mathbb{T} \rightarrow \mathcal{O}_f$ are Zariski dense in $\text{Spec}(\mathbb{T})$ under mild hypotheses, thanks to the recent results of Emerton and Kisin eluded to above. Thus the assertion (1.1) should characterize the ideal $\mathcal{L}(\kappa)$ of the Kolyvagin constructed p -adic L -function.

The paper is organized as follows. After setting our notation, we define in §2 what a Selmer structure is. In §3, we state our main technical result (Theorem 3.12) and prove it modulo the existence of core vertices. The existence of core vertices then is proved in §4. Finally in §5, we discuss several applications of our result. Bounds on Selmer groups in terms of the big Kolyvagin system we prove to exist is obtained in §5.3 and §5.4. More concrete applications towards the interpolation of Kato's Kolyvagin systems for elliptic modular forms is discussed in §5.1, §5.2 and §5.4.2.

1.1. Notations. For any field K , fix a separable closure \bar{K} of K and set $G_K = \text{Gal}(\bar{K}/K)$. Let F be a number field and λ be a non-archimedean place of F . Fix a decomposition subgroup $G_\lambda < G_F$ and let $I_\lambda < G_\lambda$ denote the inertia subgroup. Often we will identify G_λ by G_{F_λ} . For a finite set Σ of places of K , define K_Σ to be the maximal extension of K unramified outside Σ .

Let p be an odd prime and let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic \mathbb{Z}_p -extension. Let μ_{p^n} denote the p^n -th roots of unity and set $\mu_{p^\infty} = \varinjlim \mu_{p^n}$. Set $\Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and fix a topological generator γ of Γ . Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra.

For a ring S , an S -module M and an ideal I of S , let $M[I]$ denote the submodule of M consisting of elements that are killed by I .

For the ring $R = \mathcal{O}[[X_1, X_2, X_3]]$, we set $R_{u,v,w} := R/(X_1^u, X_2^v, X_3^w)$ and $R_{r,u,v,w} := R/(\varpi^r, X_1^u, X_2^v, X_3^w)$. We define the quotient modules $\mathbb{T}_{u,v,w} := \mathbb{T} \otimes_R R_{u,v,w}$ and $\mathbb{T}_{r,u,v,w} := \mathbb{T} \otimes_R R_{r,u,v,w}$.

Similarly for the ring $\mathfrak{R} = \mathcal{R}[[\Gamma]]$ as above, define the rings $\mathfrak{R}_{u,v} = \mathfrak{R}/(X^u, (\gamma - 1)^v)$ and $\mathfrak{R}_{r,u,v} = \mathfrak{R}/(\varpi^r, X^u, (\gamma - 1)^v)$. Define also the quotient modules $\mathfrak{T}_{u,v} := \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{u,v}$ and $\mathfrak{T}_{r,u,v} := \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{r,u,v}$.

Finally, we define the p -divisible groups $\mathcal{A}_{u,v,w} := \mathbb{T}_{u,v,w} \otimes \Phi/\mathcal{O}$ and $\mathfrak{A}_{u,v} := \mathfrak{T}_{u,v} \otimes \Phi/\mathcal{O}$.

Let $\mathcal{R}_0 = \mathcal{R}/(\varpi, X)$ be the dimension-zero Gorenstein artinian ring, where \mathcal{R} is as above. As explained in [Til97, Proposition 1.4],

$$(1.2) \quad \mathcal{R}_0[\mathfrak{m}_{\mathcal{R}}] \text{ is a one-dimensional } \mathfrak{k} = \mathcal{R}/\mathfrak{m}_{\mathcal{R}}\text{-vector space}$$

where $\mathfrak{m}_{\mathcal{R}}$ denotes the maximal ideal of \mathcal{R} . Define also $\mathcal{R}_1 = \mathcal{R}/X$. Using the fact $\{\varpi, X\}$ is a regular sequence in \mathcal{R} , we see that \mathcal{R}_1 is a dimension-1 Gorenstein domain. Set $\tilde{\Phi} = \text{Frac}(\mathcal{R}_1)$. As \mathcal{R}_1 is finitely generated and free as an \mathcal{O} -module, it follows that $\tilde{\Phi}$ is a finite extension of Φ . Let \mathfrak{D} be the integral closure of \mathcal{R}_1 in $\tilde{\Phi}$. Then \mathfrak{D} is a discrete valuation ring and $\mathfrak{D}/\mathcal{R}_1$ has finite cardinality. Let $\mathfrak{m}_{\mathfrak{D}}$ be the maximal ideal of \mathfrak{D} and $\pi_{\mathfrak{D}}$ be a uniformizer of \mathfrak{D} . Define $T_{\mathfrak{D}} := \mathfrak{T}_{1,1} \otimes_{\mathcal{R}_1} \mathfrak{D}$ (deformation of \bar{T} to \mathfrak{D}) and $A = T_{\mathfrak{D}} \otimes \mathbb{Q}_p/\mathbb{Z}_p$. As $\mathfrak{D}/\mathcal{R}_1$ is of finite order, it follows that $A \cong \mathfrak{A}_{1,1}$.

2. LOCAL CONDITIONS AND SELMER GROUPS

We recall a definition from [MR04, §2]. Let M be any $\mathcal{O}[[G_{\mathbb{Q}}]]$ -module.

Definition 2.1. A Selmer structure \mathcal{F} on M is a collection of the following data:

- A finite set $\Sigma(\mathcal{F})$ of places of \mathbb{Q} , including ∞ , p , and all primes where M is ramified.
- For every $\ell \in \Sigma(\mathcal{F})$, a local condition on M (which we now view as a $\mathcal{O}[[G_{\ell}]]$ -module), i.e., a choice of an \mathcal{O} -submodule $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, M) \subset H^1(\mathbb{Q}_{\ell}, M)$.

Definition 2.2. For a Selmer structure \mathcal{F} on M , define the Selmer group $H_{\mathcal{F}}^1(\mathbb{Q}, M)$ to be

$$H_{\mathcal{F}}^1(\mathbb{Q}, M) = \ker \left(H^1(\mathbb{Q}_{\Sigma(\mathcal{F})}/\mathbb{Q}, M) \longrightarrow \prod_{\ell \in \Sigma(\mathcal{F})} \frac{H^1(\mathbb{Q}_{\ell}, M)}{H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, M)} \right).$$

Definition 2.3. A Selmer triple is a triple $(M, \mathcal{F}, \mathcal{P})$ where \mathcal{F} is a Selmer structure on M and \mathcal{P} is a set of rational primes, disjoint from $\Sigma(\mathcal{F})$.

Definition 2.4. Let \mathcal{F} be a Selmer structure on M . For each prime $\ell \in \Sigma(\mathcal{F})$, define $H_{\mathcal{F}^*}^1(\mathbb{Q}_\ell, M^*) := H_{\mathcal{F}}^1(\mathbb{Q}_\ell, M)^\perp$ as the orthogonal complement of $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, M)$ under the local Tate pairing. The Selmer structure \mathcal{F}^* on M^* defined in this manner is called the *dual Selmer structure*.

Define the Selmer structure \mathcal{F}_{can} (the *canonical Selmer structure*) on $\mathbb{T}_{u,v,w}$ as follows:

- $\Sigma(\mathcal{F}_{\text{can}}) = \{\ell : \mathbb{T} \text{ is ramified at } \ell\} \cup \{p, \infty\}$.
- $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) := \begin{cases} H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) & , \text{ if } \ell = p, \\ H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) & , \text{ if } \ell \in \Sigma(\mathcal{F}_{\text{can}}) - \{p, \infty\}. \end{cases}$

Here

$$H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) := \ker \{ H^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathbb{T}_{u,v,w} \otimes_{\mathcal{O}} \Phi) \}.$$

We denote the Selmer structure on the quotients $\mathbb{T}_{r,u,v,w}$ obtained by *propagating* \mathcal{F}_{can} on $\mathbb{T}_{u,v,w}$ to $\mathbb{T}_{r,u,v,w}$ also by \mathcal{F}_{can} . See [MR04, Example 1.1.2] for a definition of the *propagation* of local conditions.

Similarly, define the Selmer structure \mathcal{F}_{can} on $\mathfrak{T}_{u,v}$ by setting

- $\Sigma(\mathcal{F}_{\text{can}}) = \{\ell : \mathfrak{T} \text{ is ramified at } \ell\} \cup \{p, \infty\}$.
- $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{u,v}) := \begin{cases} H^1(\mathbb{Q}_p, \mathfrak{T}_{u,v}) & , \text{ if } \ell = p, \\ H_f^1(\mathbb{Q}_\ell, \mathfrak{T}_{u,v}) & , \text{ if } \ell \in \Sigma(\mathcal{F}_{\text{can}}) - \{p, \infty\} \end{cases}$,

where

$$H_f^1(\mathbb{Q}_\ell, \mathfrak{T}_{u,v}) = \ker \{ H^1(\mathbb{Q}_\ell, \mathfrak{T}_{u,v}) \longrightarrow H^1(I_\ell, \mathfrak{T}_{u,v} \otimes_{\mathcal{O}} \Phi) \}.$$

Similarly, the Selmer structure on the quotients $\mathfrak{T}_{r,u,v}$ obtained by propagating \mathcal{F}_{can} on $\mathfrak{T}_{u,v}$ to $\mathfrak{T}_{r,u,v}$ is also denoted by \mathcal{F}_{can} . We also define a Selmer structure \mathcal{F}_{can} on $\mathfrak{T}/\mathfrak{m}$ by

- setting $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_p, \mathfrak{T}/\mathfrak{m}) = H^1(\mathbb{Q}_p, \mathfrak{T}/\mathfrak{m})$ and
- propagating the local conditions at $\ell \neq p$ given by \mathcal{F}_{can} on $\mathfrak{T}_{1,1}$ to $\mathfrak{T}/\mathfrak{m}$.

Note in particular that the Selmer structure \mathcal{F}_{can} on $\mathfrak{T}/\mathfrak{m}$ will not always be the propagation of the canonical Selmer structure $\mathfrak{T}_{1,1}$.

2.1. Local conditions at $\ell \neq p$. In this section we compare various alterations of the local conditions at $\ell \neq p$. Define

$$H_{\text{ur}}^1(\mathbb{Q}_\ell, M) := \ker (H^1(\mathbb{Q}_\ell, M) \longrightarrow H^1(I_\ell, M))$$

for any M on which G_ℓ acts. Using the exact sequence

$$0 \longrightarrow \mathbb{T}_{u,v,w} \longrightarrow \mathbb{T}_{u,v,w} \otimes_{\mathcal{O}} \Phi \longrightarrow \mathcal{A}_{u,v,w} \longrightarrow 0$$

define also

$$H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = \text{im} (H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w} \otimes_{\mathcal{O}} \Phi) \longrightarrow \mathcal{A}_{u,v,w}).$$

Define finally

$$H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})} \right),$$

where the map is induced from the injection $\mathbb{T}_{r,u,v,w} \hookrightarrow \mathcal{A}_{u,v,w}$. Lemma 1.3.8(i) of [Rub00] shows that

$$(2.1) \quad H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}).$$

Similarly one defines $H_f^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})$ and $H_f^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v})$, and verifies using [Rub00, Lemma 1.3.8(i)] that

$$(2.2) \quad H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) = H_f^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}).$$

2.2. Local conditions at p .

Proposition 2.5. *Assuming (H.nA),*

- (i) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w}) = H^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w})$,
- (ii) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) = H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})$.

Proof. We need to check that

$$\text{coker}(H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w})) = 0.$$

Note that

$$\text{coker}(H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w})) = H^2(\mathbb{Q}_p, \mathbb{T}_{u,v,w})[\varpi^r],$$

so it suffices to check that $H^2(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) = 0$.

Now by local duality, (H.nA) implies that $H^2(\mathbb{Q}_p, \bar{T}) = 0$. Using the fact that the cohomological dimension of G_p is two, it follows that

$$H^2(\mathbb{Q}_p, \mathbb{T})/\mathcal{M} \cdot H^2(\mathbb{Q}_p, \mathbb{T}) = 0,$$

where $\mathcal{M} = (\varpi, X_1, X_2, X_3)$ is the maximal ideal of the ring R . By Nakayama's Lemma, we therefore see that $H^2(\mathbb{Q}_p, \mathbb{T}) = 0$. Using again the fact that the cohomological dimension of G_p is two, we conclude that

$$0 = H^2(\mathbb{Q}_p, \mathbb{T}) \twoheadrightarrow H^2(\mathbb{Q}_p, \mathbb{T}_{u,v,w})$$

and the proof of (i) follows.

The proof of (ii) is similar but more delicate as the ring \mathfrak{R} is not necessarily regular. As above, we first check that

$$(2.3) \quad H^2(\mathbb{Q}_p, \mathfrak{T}) = 0.$$

Considering the G_p -cohomology induced from the exact sequence

$$0 \longrightarrow \mathfrak{T} \xrightarrow{\gamma-1} \mathfrak{T} \longrightarrow \mathcal{T} \longrightarrow 0$$

and using Nakayama's lemma, (2.3) is reduced to verifying that $H^2(\mathbb{Q}_p, \mathcal{T}) = 0$. Similarly, using the exact sequences

$$0 \longrightarrow \mathcal{T} \xrightarrow{X} \mathcal{T} \longrightarrow \mathcal{T}/X \longrightarrow 0$$

$$0 \longrightarrow \mathcal{T}/X \xrightarrow{\varpi} \mathcal{T}/X \longrightarrow \mathfrak{T}_{1,1} \longrightarrow 0$$

in turn, we reduce to checking that

$$(2.4) \quad H^2(\mathbb{Q}_p, \mathfrak{T}_{1,1,1}) = 0$$

The assertion (2.4) is proved below. We first show that (ii) follows from (2.4).

As above,

$$\text{coker}(H^1(\mathbb{Q}_p, \mathfrak{T}_{u,v}) \longrightarrow H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})) = H^2(\mathbb{Q}_p, \mathfrak{T}_{u,v})[\varpi^r].$$

By (2.4) and the fact that the cohomological dimension of G_p is two, it follows that

$$0 = H^2(\mathbb{Q}_p, \mathfrak{T}) \twoheadrightarrow H^2(\mathbb{Q}_p, \mathfrak{T}_{u,v}).$$

This proves that $H^2(\mathbb{Q}_p, \mathfrak{T}_{u,v}) = 0$ and it follows that

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) = \text{im} \left(H^1(\mathbb{Q}_p, \mathfrak{T}_{u,v}) \longrightarrow H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) \right) = H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}),$$

as desired. \square

Claim. Assuming **(H.nA)**, we have $H^2(\mathbb{Q}_p, \mathfrak{T}_{1,1,1}) = 0$.

Proof. The property (1.2) shows that $\mathfrak{T}_{1,1,1}[\mathfrak{m}] \cong \bar{T}$, hence that $\mathfrak{T}_{1,1,1}^*/\mathfrak{m} \cong \bar{T}^*$. Since we assumed **H.nA**, it thus follows that

$$H^0(\mathbb{Q}_p, \mathfrak{T}_{1,1,1}^*/\mathfrak{m}) = 0.$$

The module $\mathfrak{T}_{1,1,1}^*$ is free of finite rank over the Gorenstein artinian ring \mathcal{R}_0 , hence by [MR04, Lemma 2.1.4] we conclude that $H^0(\mathbb{Q}_p, \mathfrak{T}_{1,1,1}^*) = 0$ as well. Claim now follows by local duality. \square

2.3. Kolyvagin primes and transverse conditions. Let $\tau \in G_{\mathbb{Q}}$ be as in the statement of the hypothesis **(H.2)**.

Definition 2.6. For $\bar{n} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$, define

- (i) $H_{\bar{n}} = \ker(G_{\mathbb{Q}} \rightarrow \text{Aut}(\mathbb{T}_{r,u,v,w}) \oplus \text{Aut}(\mu_{p^r}))$,
- (ii) $L_{\bar{n}} = \bar{\mathbb{Q}}^{H_{\bar{n}}}$,
- (iii) $\mathcal{P}_{\bar{n}} = \{\text{primes } \ell : \text{Fr}_{\ell} \text{ is conjugate to } \tau \text{ in } \text{Gal}(L_{\bar{n}}/\mathbb{Q})\}$.

The collection $\mathcal{P}_{\bar{n}}$ is called the collection of *Kolyvagin primes* for $\mathbb{T}_{r,u,v,w}$. Set $\mathcal{P} = \mathcal{P}_{(1,1,1,1)}$ and define $\mathcal{N}_{\bar{n}}$ to be the set of square free products of primes in $\mathcal{P}_{\bar{n}}$.

We similarly define for $\bar{s} = (r, u, v)$ the collection of Kolyvagin primes $\mathcal{P}_{\bar{s}}$ for $\mathfrak{T}_{r,u,v}$ and the set $\mathcal{N}_{\bar{s}}$ of square free products of primes in $\mathcal{P}_{\bar{s}}$.

Definition 2.7. The partial order \prec on the collection of quadruples $(r, u, v, w) \in (\mathbb{Z}_{>0})^4$ is defined by setting

$$\bar{n} = (r, u, v, w) \prec (r', u', v', w') = \bar{n}'$$

if $r \leq r', u \leq u', v \leq v'$ and $w \leq w'$.

We denote the partial order defined on triples of positive integers in an identical manner also by \prec .

To ease notation, set $\mathbb{T}_{\bar{n}} := \mathbb{T}_{r,u,v,w}$ and $R_{\bar{n}} := R_{r,u,v,w}$ for $\bar{n} = (r, u, v, w)$. Define similarly $\mathfrak{T}_{\bar{n}} := \mathfrak{T}_{r,u,v}$ and $\mathfrak{R}_{\bar{n}} := \mathfrak{R}_{r,u,v}$.

Remark 2.8. Suppose ℓ is a Kolyvagin prime in $\mathcal{P}_{\bar{n}}$ (resp., in $\mathcal{P}_{\bar{s}}$), where \bar{n} (resp., \bar{s}) are as above. Then as τ acts trivially on μ_{p^r} and Fr_{ℓ} is conjugate to τ in $\text{Gal}(L_{\bar{n}}/\mathbb{Q})$, it follows that Fr_{ℓ} acts trivially on μ_{p^r} and hence that $\ell \equiv 1 \pmod{p^r}$. In particular

$$\begin{aligned} |\mathbb{F}_{\ell}^{\times}| \cdot \mathbb{T}_{r,u,v,w} &= (\ell - 1)\mathbb{T}_{r,u,v,w} = 0 \\ (\text{resp., } |\mathbb{F}_{\ell}^{\times}| \cdot \mathfrak{T}_{r,u,v} &= 0). \end{aligned}$$

Throughout this section, fix a Kolyvagin prime $\ell \in \mathcal{P}_{\bar{n}}$ (or in $\mathcal{P}_{\bar{s}}$, whenever we talk about quotients of \mathfrak{T}).

Definition 2.9. Let T be one of $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$.

(i) The submodule of $H^1(\mathbb{Q}_\ell, T)$ given by

$$H_{\text{tr}}^1(\mathbb{Q}_\ell, T) = \ker (H^1(\mathbb{Q}_\ell, T) \longrightarrow H^1(\mathbb{Q}_\ell(\boldsymbol{\mu}_\ell), T))$$

is called the *transverse submodule*.

(ii) The *singular quotient* $H_s^1(\mathbb{Q}_\ell, T)$ is defined by the exactness of the sequence

$$(2.5) \quad 0 \longrightarrow H_f^1(\mathbb{Q}_\ell, T) \longrightarrow H^1(\mathbb{Q}_\ell, T) \longrightarrow H_s^1(\mathbb{Q}_\ell, T) \longrightarrow 0$$

Definition 2.10. Let T be one of $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$ and suppose $n \in \mathcal{N}_{\bar{n}}$ (or $n \in \mathcal{N}_{\bar{s}}$ if we are talking about quotients of \mathfrak{T}). The *modified Selmer structure* $\mathcal{F}_{\text{can}}(n)$ on T is defined with the following data:

- $\Sigma(\mathcal{F}_{\text{can}}(n)) = \Sigma(\mathcal{F}_{\text{can}}) \cup \{\text{primes } \ell : \ell \mid n\}$.
- If $\ell \nmid n$ then $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}_\ell, T) = H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, T)$.
- If $\ell \mid n$ then $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}_\ell, T) = H_{\text{tr}}^1(\mathbb{Q}_\ell, T)$.

Remark 2.11. Proposition 1.3.2 of [MR04] shows that $\mathcal{F}_{\text{can}}(n)^* = \mathcal{F}_{\text{can}}^*(n)$.

Lemma 2.12. Let T be one of the rings $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$. Then the transverse subgroup $H_{\text{tr}}^1(\mathbb{Q}_\ell, T) \subset H^1(\mathbb{Q}_\ell, T)$ projects isomorphically onto $H_s^1(\mathbb{Q}_\ell, T)$. In other words (2.5) above has a functorial splitting.

Proof. This is [MR04, Lemma 1.2.4] which is proved for a general artinian coefficient ring. \square

Proposition 2.13. Let $\bar{n} = (r, u, v, w)$ and $\bar{s} = (r, u, v)$ be as above.

(i) There are canonical functorial isomorphisms

$$H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) \cong \mathbb{T}_{\bar{n}}/(\text{Fr}_\ell - 1)\mathbb{T}_{\bar{n}},$$

$$H_s^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) \cong (\mathbb{T}_{\bar{n}})^{\text{Fr}_\ell=1}.$$

(ii) There is a canonical isomorphism (called the *finite-singular comparison isomorphism*)

$$\phi_\ell^{fs} : H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) \longrightarrow H_s^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) \otimes \mathbb{F}_\ell^\times.$$

(iii) The $R_{\bar{n}}$ modules $H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}})$, $H_s^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}})$ and $H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}})$ are free of rank one.

The analogous statements hold true when $\mathbb{T}_{\bar{n}}$ is replaced by $\mathfrak{T}_{\bar{s}}$ or $\mathfrak{T}/\mathfrak{m}$ (and the ring $R_{\bar{n}}$ by $\mathfrak{R}_{\bar{s}}$ or $k = \mathfrak{R}/\mathfrak{m}$).

Proof. (i) is [MR04, Lemma 1.2.1]. The finite-singular comparison isomorphism is defined in [MR04, Definition 1.2.2] and (ii) is [MR04, Lemma 1.2.3]. (iii) follows from (i), (ii) and Lemma 2.12.

Note that all the results quoted from [MR04] apply in our setting thanks to Remark 2.8. \square

3. CORE VERTICES AND DEFORMING KOLYVAGIN SYSTEMS

Let $\bar{n} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$ and $\bar{s} = (r, u, v) \in (\mathbb{Z}_{>0})^3$. Assume throughout this section that $\chi(\mathbb{T}) = \chi(\mathfrak{T}) = 1$.

3.1. Core vertices. Suppose T is one of $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$ and S is the corresponding quotient ring $R_{r,u,v,w}$, $\mathfrak{R}_{r,u,v}$ or \mathbf{k} .

Definition 3.1. The integer $n \in \mathcal{N}_{\bar{n}}$ (resp., $n \in \mathcal{N}_{\bar{s}}$) is called a *core vertex* for the Selmer structure \mathcal{F}_{can} on T if

- (i) $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T^*) = 0$,
- (ii) $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T)$ is a free S -module of rank one.

Suppose that the hypotheses **(H1)**–**(H4)**, **(H.Tam)**, and **(H.nA)** hold true. The following theorem is fundamental in proving the existence of Kolyvagin systems.

Theorem 3.2. Let $n \in \mathcal{N}_{\bar{n}}$ (resp., $n \in \mathcal{N}_{\bar{s}}$) be a core vertex for the Selmer structure \mathcal{F}_{can} on the residual representation \bar{T} . Then n is a core vertex for the Selmer structure \mathcal{F}_{can} on T as well.

Theorem 3.2 is proved in §4.2. We first show how Theorem 3.2 is used to prove the existence of Kolyvagin systems for the big Galois representations \mathbb{T} (resp., for \mathfrak{T}).

3.2. Kolyvagin systems for the big Galois representation.

3.2.1. Kolyvagin systems over Artinian rings. Throughout this section fix $\bar{n} = (r, u, v, w)$ and $\bar{i} \in (\mathbb{Z}_{>0})^4$ (resp., $\bar{s} = (r, u, v)$ and $\bar{j} \in (\mathbb{Z}_{>0})^3$) such that $\bar{n} \prec \bar{i}$ (resp., $\bar{s} \prec \bar{j}$).

Throughout this subsection, let T be one of $\mathbb{T}_{\bar{n}}$, $\mathfrak{T}_{\bar{s}}$ or $\mathfrak{T}/\mathfrak{m}$, and let S be the corresponding quotient ring $R_{\bar{n}}$, $\mathfrak{R}_{\bar{s}}$ or \mathbf{k} . Let \mathcal{P} denote the collection of Kolyvagin primes $\mathcal{P}_{\bar{i}}$ (resp., $\mathcal{P}_{\bar{j}}$) and \mathcal{N} denote the set of square free products of primes in \mathcal{P} .

Much of the definitions and arguments in this section follow [MR04] and [Büy11b].

Definition 3.3.

- (i) If X is a graph and Mod_R is the category of R -modules, a *simplicial sheaf* \mathcal{S} on X with values in Mod_R is a rule assigning
 - an R -module $\mathcal{S}(v)$ for every vertex v of X ,
 - an R -module $\mathcal{S}(e)$ for every edge e of X ,
 - an R -module homomorphism $\psi_v^e : \mathcal{S}(v) \rightarrow \mathcal{S}(e)$ whenever the vertex v is an endpoint of the edge e .
- (ii) A *global section* of \mathcal{S} is a collection $\{\kappa_v \in \mathcal{S}(v) : v \text{ is a vertex of } X\}$ such that, for every edge $e = \{v, v'\}$ of X , we have $\psi_v^e(\kappa_v) = \psi_{v'}^e(\kappa_{v'})$ in $\mathcal{S}(e)$. We write $\Gamma(\mathcal{S})$ for the R -module of global sections of \mathcal{S} .

Definition 3.4. For the Selmer triple $(T, \mathcal{F}_{\text{can}}, \mathcal{P})$, we define a graph $\mathcal{X} = \mathcal{X}(\mathcal{P})$ by taking the set of vertices of \mathcal{X} to be \mathcal{N} , and the edges to be $\{n, n\ell\}$ whenever $n, n\ell \in \mathcal{N}$ (with ℓ prime).

- (i) The *Selmer sheaf* \mathcal{H} is the simplicial sheaf on \mathcal{X} given as follows. Set $G_n := \bigotimes_{\ell|n} \mathbb{F}_{\ell}^{\times}$. We take

- $\mathcal{H}(n) := H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T) \otimes G_n$ for $n \in \mathcal{N}$,
- if e is the edge $\{n, n\ell\}$ then $\mathcal{H}(e) := H_s^1(\mathbb{Q}_{\ell}, T) \otimes G_{n\ell}$.

We define the vertex-to-edge maps to be

- $\psi_{n\ell}^e : H_{\mathcal{F}_{\text{can}}(n\ell)}^1(\mathbb{Q}, T) \otimes G_{n\ell} \rightarrow H_s^1(\mathbb{Q}_{\ell}, T) \otimes G_{n\ell}$ is localization followed by the projection to the singular cohomology $H_s^1(\mathbb{Q}_{\ell}, T)$.
- $\psi_n^e : H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T) \otimes G_n \rightarrow H_s^1(\mathbb{Q}_{\ell}, T) \otimes G_{n\ell}$ is the composition of localization at ℓ with the finite-singular comparison map ϕ_{ℓ}^{fs} .

- (ii) A *Kolyvagin system* for the triple $(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ is simply a global section of the Selmer sheaf \mathcal{H} .

We write $\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) := \Gamma(\mathcal{H})$ for the S -module of Kolyvagin systems for the Selmer structure \mathcal{F}_{can} on T . More explicitly, an element $\kappa \in \mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ is a collection $\{\kappa_n\}$ of cohomology classes indexed by $n \in \mathcal{N}$ such that for every $n, n\ell \in \mathcal{N}$ we have:

- $\kappa_n \in H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T) \otimes G_n$,
- $\phi_\ell^{fs}(\text{loc}_\ell(\kappa_n)) = \text{loc}_\ell^s(\kappa_{n\ell})$.

Here, loc_ℓ^s stands for the composite

$$H^1(\mathbb{Q}, T) \xrightarrow{\text{loc}_\ell} H^1(\mathbb{Q}_\ell, T) \longrightarrow H_s^1(\mathbb{Q}_\ell, T).$$

The goal of this section is to prove the following theorem, assuming **(H1)**–**(H4)**, **(H.Tam)** and **(H.nA)**.

Theorem 3.5. *The S -module $\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one.*

Theorem 3.5 is proved in two steps. As the first step, we prove:

Theorem 3.6. *Suppose $n \in \mathcal{N}$ is any core vertex for the Selmer structure \mathcal{F}_{can} on T . Then the natural map*

$$\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \longrightarrow H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T) \otimes G_n$$

(given by $\kappa \mapsto \kappa_n$) is surjective.

The arguments of [Büy11b, Theorem 3.11] (which in turn modifies the arguments of Howard in [MR04] appropriately so as to apply them with general artinian rings) may be used to prove Theorem 3.6. The main point is that we have Theorem 3.2 here in place of [Büy11b, Theorem 2.27].

Define a subgraph $\mathcal{X}^0 = \mathcal{X}^0(\mathcal{P})$ of \mathcal{X} whose vertices are the core vertices of \mathcal{X} and whose edges are defined as follows: We join n and $n\ell$ by an edge in \mathcal{X}^0 if and only if the localization map

$$H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \bar{T}) \longrightarrow H_f^1(\mathbb{Q}_\ell, \bar{T})$$

is *non-zero*. We define the sheaf \mathcal{H}^0 on \mathcal{X}^0 as the restriction of the Selmer sheaf \mathcal{H} to \mathcal{X}^0 .

Lemma 3.7. *The graph \mathcal{X}^0 is connected.*

Proof. The edges of \mathcal{X}^0 are defined in terms of \bar{T} (and not T itself) so [MR04, Theorem 4.3.12] applies. \square

The following Theorem, combined with Theorem 3.6 completes the proof of Theorem 3.5.

Theorem 3.8. *Suppose $n \in \mathcal{N}$ is any core vertex for the Selmer structure \mathcal{F}_{can} on T . Then the natural map*

$$\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \longrightarrow H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, T) \otimes G_n$$

is injective.

Theorem 3.5 is proved using the arguments that goes into the proof of [Büy11b, Theorem 3.12]. The essential input is the fact that the graph \mathcal{X}^0 is connected (Lemma 3.7).

3.2.2. *Kolyvagin systems over ‘big’ rings.* The goal of this section is to prove (Theorem 3.12 below) using the results of the previous section that the R -module

$$\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) := \varprojlim_{\bar{n}} \left(\varinjlim_{\bar{i}} \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{i}}) \right)$$

(resp., the \mathfrak{R} -module

$$\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) := \varprojlim_{\bar{s}} \left(\varinjlim_{\bar{j}} \mathbf{KS}(\mathfrak{T}_{\bar{s}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{j}}) \right)$$

is free of rank one.

Lemma 3.9. *Fix a quadruple \bar{n} as above. For any $\bar{i} \succ \bar{n}$, the natural restriction map*

$$\mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}) \longrightarrow \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{i}})$$

is an isomorphism.

Proof. Theorems 3.6 and 3.8 applied with a core vertex $n \in \mathcal{N}_{\bar{i}}$, we have isomorphisms

$$\mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}) \xrightarrow{\sim} H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}) \xleftarrow{\sim} \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{i}})$$

compatible with the restriction map

$$\mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}) \longrightarrow \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{i}}).$$

Note that $n \in \mathcal{N}_{\bar{i}}$ as above exists by [MR04, Corollary 4.1.9] and Theorem 3.2 above. \square

Lemma 3.10. *Let $\bar{n}' \prec \bar{n}$ and let $n \in \mathcal{N}_{\bar{n}}$ be a core vertex. The map*

$$H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}) \longrightarrow H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}'})$$

is surjective.

Proof. We verify the assertion of the Lemma for $\bar{n}' = (r, u, v, w)$ and $\bar{n} = (r+1, u, v, w)$. The proof of the general case follows by applying this argument (or where necessary, its slightly modified form) repeatedly.

We have the following commutative diagram, where the vertical isomorphism is obtained from (a slight variation of) Proposition 4.17(iv) below:

$$\begin{array}{ccc} H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{r+1, u, v, w}) & \xrightarrow{\text{reduction}} & H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{r, u, v, w}) \\ & \searrow \varpi & \cong \downarrow [\varpi] \\ & & H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{r+1, u, v, w})[\varpi^r] \end{array}$$

Since $n \in \mathcal{N}_{\bar{n}}$ is a core vertex (and therefore $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}})$ is a free $R_{\bar{n}}$ -module of rank one), the map on the diagonal is surjective. This proves that the horizontal map is surjective as well. \square

Lemma 3.11. *The map*

$$\mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}) \longrightarrow \mathbf{KS}(\mathbb{T}_{\bar{n}'}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}})$$

is surjective for $\bar{n}' \prec \bar{n}$.

Proof. By Theorem 3.6, Theorem 3.8 and Lemma 3.9 applied with a core vertex $n \in \mathcal{N}_{\bar{n}}$ to both $\mathbb{T}_{\bar{n}}$ and $\mathbb{T}_{\bar{n}'}$, we obtain the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}) & \longrightarrow & \mathbf{KS}(\mathbb{T}_{\bar{n}'}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}) \\ \cong \downarrow & & \downarrow \cong \\ H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}) \otimes G_n & \twoheadrightarrow & H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}'}) \otimes G_n \end{array}$$

where the surjection in the second row is Lemma 3.10. It follows at once that the upper horizontal map in the diagram is surjective as well. \square

Theorem 3.12. *Under the running hypotheses the following hold.*

- (i) *The R -module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one, generated by a Kolyvagin system κ whose image $\bar{\kappa} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero.*
- (ii) *The \mathfrak{R} -module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one. When the ring \mathcal{R} is regular, the module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is generated by κ whose image $\bar{\kappa} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero.*

Proof. Lemma 3.9 shows that

$$\varinjlim_{\bar{i}} \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{i}}) = \mathbf{KS}(\mathbb{T}_{\bar{n}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{n}}).$$

The proof of (i) now follows by Theorem 3.5 and Lemma 3.11. (ii) is proved similarly, by appropriately modifying the ingredients that go into the proof of (i). \square

4. THE EXISTENCE OF CORE VERTICES

The goal of this section is to verify the truth of Theorem 3.2.

4.1. Cartesian properties. Let $\mathcal{C} = \{\mathbb{T}_{r,u,v,w} : r, u, v, w \in \mathbb{Z}^+\}$ (resp., $\mathfrak{C} = \{\mathfrak{T}_{r,u,v} : r, u, v \in \mathbb{Z}^+\} \cup \{\mathfrak{T}/\mathfrak{m}\}$) be a collection of R -modules (resp., \mathfrak{R} -modules).

Definition 4.1. A local condition \mathcal{F} at a prime ℓ is said to be *cartesian* on the collection \mathcal{C} if it satisfies the following conditions:

(C1) (weak Functoriality) If $r \leq r', u \leq u', v \leq v', w \leq w'$ are positive integers, then

(C1.a) $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w})$ is the exact image of $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r',u,v,w})$ under the canonical map

$$H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r',u,v,w}) \longrightarrow H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w}),$$

(C1.b) $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w})$ lies inside the image of $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u',v,w})$ under the canonical map

$$H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u',v,w}) \longrightarrow H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w}).$$

(C1.c) $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w})$ lies inside the image of $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w'})$ under the canonical map

$$H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w'}) \longrightarrow H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w}).$$

(C1.d) $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w})$ lies inside the image of $H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w'})$ under the canonical map

$$H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w'}) \longrightarrow H^1(\mathbb{Q}_{\ell}, \mathbb{T}_{r,u,v,w}).$$

(C2) (Cartesian property in X_i -direction) If $u, v, w, r \in \mathbb{Z}^+$, then

$$(C2.a) \quad H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u+1,v,w})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u+1,v,w})} \right).$$

The arrow is induced from the injection

$$\mathbb{T}_{r,u,v,w} \xrightarrow{[X_1]} \mathbb{T}_{r,u+1,v,w}$$

where $[X_1]$ stands for multiplication by X_1 .

$$(C2.b) \quad H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \xrightarrow{[X_2]} \frac{H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v+1,w})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v+1,w})} \right).$$

$$(C2.c) \quad H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \xrightarrow{[X_3]} \frac{H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w+1})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w+1})} \right).$$

(C3) (Cartesian property as powers of p vary) If $r \leq r'$ are positive integers, then

$$H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \xrightarrow{[\varpi]} \frac{H^1(\mathbb{Q}_\ell, \mathbb{T}_{r+1,u,v,w})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r+1,u,v,w})} \right),$$

where the arrow is induced from the injection

$$\mathbb{T}_{r,u,v,w} \xrightarrow{[\varpi]} \mathbb{T}_{r+1,u,v,w}.$$

Similarly,

Definition 4.2. A local condition \mathcal{F} at a prime ℓ is said to be *cartesian* on the collection \mathfrak{C} if it satisfies the following conditions:

(D1) If $r \leq r', u \leq u', v \leq v'$ are positive integers, then

(D1.a) $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v})$ is the exact image of $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r',u,v})$ under the canonical map

$$H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r',u,v}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}),$$

(D1.b) $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v})$ lies inside the image of $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u',v})$ under the canonical map

$$H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u',v}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}).$$

(D1.c) $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v})$ lies inside the image of $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v'})$ under the canonical map

$$H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v'}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}).$$

(D2) If $u, v, r \in \mathbb{Z}^+$, then

$$(D2.a) \quad H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \xrightarrow{X} \frac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u+1,v})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u+1,v})} \right).$$

$$(D2.b) \quad H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \xrightarrow{[\gamma-1]} \frac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v+1})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v+1})} \right).$$

(D3) (Cartesian property as powers of p vary) If $r \leq r'$ are positive integers, then

$$H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \xrightarrow{[\varpi]} \frac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r+1,u,v})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r+1,u,v})} \right).$$

(D4) $H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) = \ker \left(H^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})}{H_{\mathcal{F}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})} \right)$, where the arrow is induced from the injection

$$\mathfrak{R}/\mathfrak{m} = \mathbf{k} \xrightarrow{\sim} \mathcal{R}_0[\mathfrak{m}] \hookrightarrow \mathcal{R}_0.$$

4.1.1. Cartesian properties at p .

Proposition 4.3. *Assuming **(H.nA)**, the local condition at p given by \mathcal{F}_{can} on the collection \mathcal{C} (resp., on the collection \mathfrak{C}) is cartesian.*

Proof. This is obvious thanks to Proposition 2.5. \square

4.1.2. Cartesian properties at bad primes $\ell \neq p$ over the coefficient ring R . Throughout this section the hypothesis **(H.Tam)** is in force.

Lemma 4.4.

- (i) $H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})$.
- (ii) $H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w})$.
- (iii) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w})$.
- (iv) *The following sequence is exact:*

$$0 \longrightarrow H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}$$

Proof. (iv) follows from (2.1).

By [Rub00, Lemma 1.3.5] we have the following two exact sequences:

$$(4.1) \quad 0 \longrightarrow H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \longrightarrow \mathcal{W}/(\text{Fr}_\ell - 1)\mathcal{W} \longrightarrow 0$$

$$(4.2) \quad 0 \longrightarrow H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \longrightarrow H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \longrightarrow \mathcal{W}^{\text{Fr}_\ell=1} \longrightarrow 0$$

where $\mathcal{W} = \mathcal{A}_{u,v,w}^{I_v}/(\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}$. In Lemma 4.5, we check under the assumption **(H.Tam)** that $\mathcal{W}^{\text{Fr}_\ell=1} = 0$. Since \mathcal{W} is a finite module, the exact sequence

$$0 \longrightarrow \mathcal{W}^{\text{Fr}_\ell=1} \longrightarrow \mathcal{W} \xrightarrow{\text{Fr}_\ell-1} \mathcal{W} \longrightarrow \mathcal{W}/(\text{Fr}_\ell - 1)\mathcal{W}$$

shows that $\mathcal{W}/(\text{Fr}_\ell - 1)\mathcal{W} = 0$. This proves that

$$(4.3) \quad H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \quad , \quad H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) = H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}).$$

This proves (i) and (ii). By (2.1) and (4.3) it now follows that

$$\begin{aligned} H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) &= \text{im} \left(H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \rightarrow H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \right) \subset H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{k,u,v,w}) \\ H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) &= \ker \left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})} \right) \supset H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathbb{T}_{k,u,v,w}) \end{aligned}$$

and the proof of (iii) follows. \square

Lemma 4.5. $\mathcal{W}^{\text{Fr}_\ell=1} = 0$.

Proof. As we have $\mathcal{A}_{1,1,1}[\varpi] = \bar{T}$, it follows that $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}[\varpi]) = 0$ since we assume **H.Tam**, hence also that

$$H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}) = 0.$$

Using the G_ℓ -cohomology of the exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{A}_{1,1,w} \xrightarrow{[X_3]} \mathcal{A}_{1,1,w+1} \longrightarrow \mathcal{A}_{1,1,1} \longrightarrow 0 \\ 0 &\longrightarrow \mathcal{A}_{1,v,w} \xrightarrow{[X_2]} \mathcal{A}_{1,v+1,w} \longrightarrow \mathcal{A}_{1,1,w} \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{A}_{u,v,w} \xrightarrow{[X_1]} \mathcal{A}_{u+1,v,w} \longrightarrow \mathcal{A}_{1,v,w} \longrightarrow 0, \end{aligned}$$

it follows by induction that

$$(4.4) \quad H^0(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = 0.$$

Taking the G_ℓ/I_ℓ -invariance of the short exact sequence

$$0 \longrightarrow (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}} \longrightarrow \mathcal{A}_{u,v,w}^{I_\ell} \longrightarrow \mathcal{W} \longrightarrow 0$$

we see by (4.4) that

$$\mathcal{W}^{\text{Fr}_\ell=1} \hookrightarrow H^1(G_\ell/I_\ell, (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}) \cong (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}/(\text{Fr}_\ell - 1).$$

To conclude with the proof, it therefore suffices to show that

$$(\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}/(\text{Fr}_\ell - 1) = 0.$$

For any $\alpha \in \mathbb{Z}^+$, (4.4) shows

$$(4.5) \quad H^0(G_\ell/I_\ell, (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha]) = 0.$$

The exact sequence

$$((\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha])^{\text{Fr}_\ell=1} \rightarrow (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha] \xrightarrow{\text{Fr}_\ell-1} (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha] \rightarrow \mathcal{A}_{u,v,w}^{I_\ell}[\varpi^\alpha]/(\text{Fr}_\ell - 1) \rightarrow 0$$

and (4.5) shows that $\mathcal{A}_{u,v,w}^{I_\ell}[\varpi^\alpha]/(\text{Fr}_\ell - 1) = 0$. Passing to direct limit the Lemma follows. \square

By Lemma 4.4(iv) we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) & \longrightarrow & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})} \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u+1,v,w}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u+1,v,w}) & \longrightarrow & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})}{H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})} \end{array}$$

Lemma 4.6. *The map α is injective if*

$$\beta : H^1(I_\ell, \mathcal{A}_{u,v,w})^{\text{Fr}_\ell=1} \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w})^{\text{Fr}_\ell=1}$$

is injective.

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})} \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w})^{\text{Fr}_\ell=1} \\ & & \downarrow \alpha \qquad \qquad \qquad \downarrow \beta \\ 0 & \longrightarrow & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})}{H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})} \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w})^{\text{Fr}_\ell=1} \end{array}$$

whose exact rows come from the Hochschild-Serre spectral sequence and the fact that

$$H_f^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) := \ker \left(H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w})^{\text{Fr}_\ell=1} \right),$$

where the first equality is Lemma 4.4(i). \square

Consider the short exact sequence

$$0 \longrightarrow \mathcal{A}_{u,v,w} \xrightarrow{[X_1]} \mathcal{A}_{u+1,v,w} \longrightarrow \mathcal{A}_{1,v,w} \longrightarrow 0$$

The I_ℓ -cohomology of this sequence gives

$$(4.6) \quad 0 \longrightarrow \mathcal{A}_{u+1,v,w}^{I_\ell} / \mathcal{A}_{u,v,w}^{I_\ell} \longrightarrow \mathcal{A}_{1,v,w}^{I_\ell} \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w})$$

To ease the notation set

$$\mathcal{K}_{v,w} = \mathcal{A}_{u+1,v,w}^{I_\ell} / \mathcal{A}_{u,v,w}^{I_\ell},$$

so that the sequence (4.6) may be rewritten as

$$(4.7) \quad 0 \longrightarrow \mathcal{A}_{1,v,w}^{I_\ell} / \mathcal{K}_{v,w} \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w})$$

Taking G_ℓ / I_ℓ -invariance in (4.7), we conclude that

Lemma 4.7. $\ker(\beta) \cong H^0(G_\ell / I_\ell, \mathcal{A}_{1,v,w}^{I_\ell} / \mathcal{K}_{v,w})$.

Lemma 4.8. *Under the assumption that (H.Tam) holds true,*

- (i) $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,v,w}) = 0$,
- (ii) $H^0(\mathbb{Q}_\ell, \mathcal{K}_{v,w}) = 0$.

Proof. Noting that $\bar{T} \cong \mathcal{A}_{1,1,1}$, Hypothesis (H.Tam) shows that

$$H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}[\varpi]) = 0$$

and also that $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}) = 0$. The G_ℓ -invariance of the sequence

$$0 \longrightarrow \mathcal{A}_{1,1,w-1} \xrightarrow{[X_3]} \mathcal{A}_{1,1,w} \longrightarrow \mathcal{A}_{1,1,1} \longrightarrow 0$$

shows by induction that $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,w}) = 0$ for all $w \in \mathbb{Z}_{\geq 2}$. Using similarly the exact sequence

$$0 \longrightarrow \mathcal{A}_{1,v-1,w} \xrightarrow{[X_2]} \mathcal{A}_{1,v,w} \longrightarrow \mathcal{A}_{1,1,w} \longrightarrow 0$$

we conclude with the proof of (i). (ii) follows from (i) as $\mathcal{K}_{v,w}$ is a submodule of $\mathcal{A}_{1,v,w}$. \square

Proposition 4.9. $\ker(\beta) = 0$.

Proof. Taking the G_ℓ / I_ℓ -invariance of the short exact sequence

$$0 \longrightarrow \mathcal{K}_{v,w} \longrightarrow \mathcal{A}_{1,v,w}^{I_\ell} \longrightarrow \mathcal{A}_{1,v,w}^{I_\ell} / \mathcal{K}_{v,w} \longrightarrow 0,$$

we conclude using Lemma 4.8 that

$$(4.8) \quad \ker(\beta) \hookrightarrow H^1(G_\ell / I_\ell, \mathcal{K}_{v,w}) \cong \mathcal{K}_{v,w} / (\text{Fr}_\ell - 1)\mathcal{K}_{v,w}.$$

Lemma 4.8(ii) yields (using the fact that $\mathcal{K}_{v,w}$ is ϖ^∞ -torsion) an exact sequence

$$0 \longrightarrow \mathcal{K}_{v,w}[\varpi^\alpha] \xrightarrow{\text{Fr}_\ell - 1} \mathcal{K}_{v,w}[\varpi^\alpha] \longrightarrow \mathcal{K}_{v,w}[\varpi^\alpha] / (\text{Fr}_\ell - 1) \longrightarrow 0$$

for every $\alpha \in \mathbb{Z}^+$. Noting that the module $\mathcal{K}_{v,w}[\varpi^\alpha]$ has finite cardinality, it follows now that

$$\mathcal{K}_{v,w}[\varpi^\alpha] / (\text{Fr}_\ell - 1) = 0.$$

Passing to direct limit, Proposition follows by (4.8). \square

Proposition 4.10. *The local condition at a prime $\ell \neq p$, given by \mathcal{F}_{can} on the collection \mathcal{C} is cartesian.*

Proof. **C1.a** is true by definition and **C1.b-d** by Lemma 4.4(iii). **C2.a-c** follow from Lemma 4.6 and Proposition 4.9; **C3** holds true thanks to [MR04, Lemma 3.7.1]. \square

4.1.3. *Cartesian properties at bad primes $\ell \neq p$ over the coefficient ring \mathfrak{A} .* Assume throughout this section that **(H.Tam)** holds true.

Lemma 4.11.

- (i) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) = H_{\text{ur}}^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})$.
- (ii) *The sequence*

$$0 \longrightarrow H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})}{H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})}$$

is exact.

Proof. The proof of Lemma 4.4 above works verbatim. □

Lemma 4.12. *In the commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) & \longrightarrow & \frac{H^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})}{H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})} \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u+1,v}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u+1,v}) & \longrightarrow & \frac{H^1(\mathbb{Q}_\ell, \mathfrak{A}_{u+1,v})}{H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{A}_{u+1,v})} \end{array}$$

the map α is injective.

Proof. Identical to the proof of Proposition 4.9. □

Recall the ring \mathfrak{D} and the module $T_{\mathfrak{D}}$ from §1.1.

Proposition 4.13.

- (i) $H_f^1(\mathbb{Q}_\ell, T_{\mathfrak{D}}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, T_{\mathfrak{D}})$, where

$$H_f^1(\mathbb{Q}_\ell, T_{\mathfrak{D}}) = \ker \left(H^1(\mathbb{Q}_\ell, T_{\mathfrak{D}}) \rightarrow H^1(I_\ell, T_{\mathfrak{D}} \otimes \mathbb{Q}_p) \right).$$
- (ii) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1})$.
- (iii) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})$.
- (iv) $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \bar{T}) = \text{im} \left(H_{\text{ur}}^1(\mathbb{Q}_\ell, T_{\mathfrak{D}}) \rightarrow H^1(\mathbb{Q}_\ell, \bar{T}) \right) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \bar{T})$.
- (v) $H_{\text{ur}}^1(\mathbb{Q}_\ell, \bar{T})$ is the inverse image of $H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1})[\mathfrak{m}]$ under the map induced from (1.2).

Proof. (i) and (ii) follows from [Rub00, Lemma 1.3.5] since we assumed **(H.Tam)**, and (iii) follows mimicking the proof of Lemma 4.4(iii). We next verify (iv). By its very definition (see the beginning of §2),

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \bar{T}) = \text{im} \left(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \rightarrow H^1(\mathbb{Q}_\ell, \bar{T}) \right) = \text{im} \left(H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \rightarrow H^1(\mathbb{Q}_\ell, \bar{T}) \right),$$

where the second equality is thanks to (i). Thus, the assertion (iv) amounts to

$$(4.9) \quad \text{im} \left(H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \rightarrow H^1(\mathbb{Q}_\ell, \bar{T}) \right) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \bar{T}),$$

$$(4.10) \quad \text{im} \left(H_{\text{ur}}^1(\mathbb{Q}_\ell, T_{\mathfrak{D}}) \rightarrow H^1(\mathbb{Q}_\ell, \bar{T}) \right) = H_{\text{ur}}^1(\mathbb{Q}_\ell, \bar{T}).$$

In order to verify (4.9), it suffices to check that we have a surjection

$$H^0(I_\ell, \mathfrak{T}_{1,1}) \twoheadrightarrow H^0(I_\ell, \bar{T})$$

as G_ℓ/I_ℓ has cohomological dimension 1. Taking the I_ℓ -invariance of the exact sequence

$$(4.11) \quad 0 \longrightarrow \mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1} \longrightarrow \mathfrak{T}_{1,1} \longrightarrow \bar{T} \longrightarrow 0$$

we see that

$$\operatorname{coker} (H^0(I_\ell, \mathfrak{T}_{1,1}) \longrightarrow H^0(I_\ell, \bar{T})) \hookrightarrow H^1(I_\ell, \mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1}).$$

As the module $H^0(I_\ell, \bar{T})$ is of finite order, the image of the injection above lands in the \mathbb{Z}_p -torsion submodule $H^1(I_\ell, \mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1})_{\text{tors}}$ of $H^1(I_\ell, \mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1})$. On the other hand,

$$H^1(I_\ell, \mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1})_{\text{tors}} \cong (\mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1} \otimes \mathbb{Q}_p / \mathbb{Z}_p)^{I_\ell} / \operatorname{div} = A^{I_\ell} / \operatorname{div} = 0$$

where

- M/div is short for M/M_{div} ;
- the second equality is obtained tensoring the exact sequence (4.11) by $\mathbb{Q}_p/\mathbb{Z}_p$ and noting that the exactness is preserved as $\mathfrak{m}_{\mathcal{R}} \mathfrak{T}_{1,1}$ is \mathbb{Z}_p -torsion free, and that $\bar{T} \otimes \mathbb{Q}_p / \mathbb{Z}_p = 0$;
- the last equality is **(H.Tam)**.

This shows that

$$\operatorname{coker} (H^0(I_\ell, \mathfrak{T}_{1,1}) \longrightarrow H^0(I_\ell, \bar{T})) = 0$$

as desired and (4.9) is verified.

To verify (4.10), it again suffices to check that

$$\operatorname{coker} (H^0(I_\ell, T_{\mathfrak{D}}) \longrightarrow H^0(I_\ell, \bar{T})).$$

Considering the I_ℓ -invariance of the exact sequence

$$0 \longrightarrow T_{\mathfrak{D}} \xrightarrow{\pi_{\mathfrak{D}}} T_{\mathfrak{D}} \longrightarrow \bar{T} \longrightarrow 0$$

we see that

$$\operatorname{coker} (H^0(I_\ell, T_{\mathfrak{D}}) \longrightarrow H^0(I_\ell, \bar{T})) \hookrightarrow H^1(I_\ell, T_{\mathfrak{D}})_{\text{tors}}.$$

As above, $H^1(I_\ell, T_{\mathfrak{D}})_{\text{tors}} \cong A^{I_\ell} / \operatorname{div} = 0$ and this completes the proof of (iv).

We now prove (v). Consider the sequence

$$(4.12) \quad 0 \longrightarrow \bar{T} \longrightarrow \mathfrak{T}_{1,1} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where the arrow $\bar{T} \rightarrow \mathfrak{T}_{1,1}$ is obtained from (1.2) and \mathcal{Q} is defined by the exactness of this sequence. Taking the I_ℓ -invariance of the sequence (4.12), we obtain another exact sequence

$$0 \longrightarrow \mathcal{Q}_0 \longrightarrow H^1(I_\ell, \bar{T}) \longrightarrow H^1(I_\ell, \mathfrak{T}_{1,1})$$

where $\mathcal{Q}_0 := \mathcal{Q}^{I_\ell} / \mathfrak{T}_{1,1}^{I_\ell} / \bar{T}^{I_\ell}$. Taking the G_ℓ / I_ℓ -invariance of the final exact sequence, we conclude that

$$\ker (H^1(I_\ell, \bar{T})^{\operatorname{Fr}_\ell=1} \longrightarrow H^1(I_\ell, \mathfrak{T}_{1,1})^{\operatorname{Fr}_\ell=1}) = \mathcal{Q}_0^{\operatorname{Fr}_\ell=1}$$

hence by Lemma 4.14 below that

$$(4.13) \quad \ker (H^1(I_\ell, \bar{T})^{\operatorname{Fr}_\ell=1} \longrightarrow H^1(I_\ell, \mathfrak{T}_{1,1})^{\operatorname{Fr}_\ell=1}) = 0.$$

Consider now the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{ur}}^1(\mathbb{Q}_\ell, \bar{T}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \bar{T}) & \longrightarrow & H^1(I_\ell, \bar{T})^{\operatorname{Fr}_\ell=1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi \\ 0 & \longrightarrow & H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) & \longrightarrow & H^1(I_\ell, \mathfrak{T}_{1,1})^{\operatorname{Fr}_\ell=1} \longrightarrow 0 \end{array}$$

(4.13) shows that φ is injective, and a simple diagram chase yields

$$H_{\text{ur}}^1(\mathbb{Q}_\ell, \bar{T}) = \ker (H^1(\mathbb{Q}_\ell, \bar{T}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) / H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}))$$

which is a restatement of (v). \square

Lemma 4.14. $\mathcal{Q}_0^{\text{Fr}_\ell=1} = 0$

Proof. As $\bar{T}^{G_\ell} = 0$, it follows by the proof of [MR04, Lemma 2.1.4] that $S^{G_\ell} = 0$ for any subquotient S of \mathfrak{T} , in particular for $S = \mathcal{Q}_0$. \square

Proposition 4.15. *The local condition at a prime $\ell \neq p$, given by \mathcal{F}_{can} on the collection \mathfrak{C} is cartesian.*

Proof. One verifies **D1** using Lemma 4.11 and **D2** using Lemma 4.12. **D3** follows from [MR04, Lemma 3.7.1] and **D4** from Proposition 4.13(i) and Proposition 4.13(iv). \square

4.1.4. *Cartesian properties for the transverse condition.* Recall the partial order \prec from Definition 2.7 on the quadruples (resp., on the triples) of positive integers.

Proposition 4.16. *For $\bar{n}_0 = (r_0, u_0, v_0, w_0)$, suppose $\ell \in \mathcal{P}_{\bar{n}}$ is a Kolyvagin prime in the sense of Definition 2.6. Then the transverse local condition at ℓ is Cartesian on the family $\{\mathbb{T}_{\bar{n}}\}_{\bar{n} \prec \bar{n}_0}$ (resp., on the family $\{\mathfrak{T}_{\bar{n}}\}_{\bar{n} \prec \bar{n}_0} \cup \{\mathfrak{T}/\mathfrak{m}\}$).*

Proof. Suppose $\bar{n} = (r, u, v, w)$ and $\bar{n}' = (r', u', v', w')$ are such that $\bar{n} \prec \bar{n}' \prec \bar{n}_0$. Then we have the following commutative diagram whose rows are exact by Lemma 2.12:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}'}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}'}) & \longrightarrow & H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}'}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) & \longrightarrow & H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}) \longrightarrow 0 \end{array}$$

where the vertical arrows are induced from the natural surjection $\mathbb{T}_{\bar{n}'} \twoheadrightarrow \mathbb{T}_{\bar{n}}$. This shows that $H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}'})$ is mapped into $H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}})$ and **C1** is therefore verified. Furthermore, as the $R_{\bar{n}'}$ -module $\mathbb{T}_{\bar{n}'}^{\text{Fr}_\ell=1}$ (resp., the $R_{\bar{n}}$ -module $\mathbb{T}_{\bar{n}}^{\text{Fr}_\ell=1}$) is free of rank one, it follows by Lemma 2.12 and Proposition 2.13(i) that

$$H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}'} \twoheadrightarrow \mathbb{T}_{\bar{n}}) \twoheadrightarrow H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}}),$$

i.e., the transverse local condition on the quotients $\mathbb{T}_{\bar{n}}$ is the same as the propagation of the local condition $H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}_0})$. As the quotient

$$H^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}_0})/H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}_0}) \cong H_f^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{n}_0})$$

is a free $R_{\bar{n}_0}$ -module of rank one, **C2** and **C3** follow from the proof of [MR04, Lemma 3.7.1(i)], using the argument in loc.cit. for the multiplication by $[X_1]$, $[X_2]$, $[X_3]$ and $[\varpi]$ maps separately.

One proves **D1-D3** for the collection $\{\mathfrak{T}_{\bar{n}}\}_{\bar{n} \prec \bar{n}_0} \cup \{\mathfrak{T}/\mathfrak{m}\}$ in an identical way and it remains to verify **D4**. To settle that, consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) & \longrightarrow & H_f^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})[\mathfrak{m}_{\mathcal{R}}] & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})[\mathfrak{m}_{\mathcal{R}}] & \longrightarrow & H_f^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})[\mathfrak{m}_{\mathcal{R}}] \longrightarrow 0 \end{array}$$

As the \mathcal{R}_0 -module $H_f^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})$ (resp., the k -vector space $H_f^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m})$) is free of rank one (resp., is one-dimensional), it follows that the right-most arrow is injective and by chasing the diagram it follows that

$$\ker \left(H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})}{H_{\text{tr}}^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})} \right),$$

which is **D4**. \square

4.2. Controlling the Selmer sheaf. Assume throughout this section that $\chi(\mathbb{T}) = \chi(\mathfrak{T}) = 1$ in addition to the running hypotheses. Let $\bar{n} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$ and $\bar{s} = (r, u, v) \in (\mathbb{Z}_{>0})^3$.

4.2.1. The upper bound.

Proposition 4.17. *We have the following isomorphisms:*

- (i) $H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, \bar{T}) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, \mathbb{T}_{\bar{n}})[\mathcal{M}],$
- (ii) $H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, \mathfrak{T}/\mathfrak{m}) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, \mathfrak{T}_{1,1,1})[\mathfrak{m}],$
- (iii) $H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, \mathfrak{T}_{1,1,1}) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, \mathfrak{T}_{\bar{s}})[(\varpi, X, \gamma - 1)],$
- (iv) $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \bar{T}) \xrightarrow{\sim} H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}})[\mathcal{M}],$
- (v) $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathfrak{T}/\mathfrak{m}) \xrightarrow{\sim} H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathfrak{T}_{\bar{s}})[\mathfrak{m}].$

Proof. (i), (ii) and (iii) follows from the proof of [MR04] Lemma 3.5.2; see in particular the displayed equation (7). (iv) is now verified using (i) and Propositions 4.3, 4.10 and 4.16. (v) follows from (ii),(iii) and Propositions 4.3, 4.15 and 4.16. \square

Let $R_{r,u,v,w} = R/(\varpi^r, X_1^u, X_2^v, X_3^w)$ and $\mathfrak{R}_{r,u,v} = \mathfrak{R}/(\varpi^r, X^u, (\gamma - 1)^v)$.

Corollary 4.18. *Let $n \in \mathcal{N}_{r,u,v,w}$ (resp., $n \in \mathcal{N}_{r,u,v}$) be a core vertex for the Selmer structure \mathcal{F}_{can} on \bar{T} , in the sense of Definition 3.1. The $R_{r,u,v,w}$ -module $\text{Hom}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{r,u,v,w}), \Phi/\mathcal{O}\right)$ (resp., the $\mathfrak{R}_{r,u,v}$ -module $\text{Hom}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathfrak{T}_{r,u,v}), \Phi/\mathcal{O}\right)$) is cyclic.*

Proof. By Proposition 4.17(iii), it follows that

$$\text{Hom}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{r,u,v,w}), \Phi/\mathcal{O}\right) / \mathcal{M} \cong \text{Hom}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \bar{T}), \Phi/\mathcal{O}\right).$$

Since the k -vector space $\text{Hom}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \bar{T}), \Phi/\mathcal{O}\right)$ is one-dimensional (thanks to our assumption that n is a core vertex and that $\chi(\mathbb{T}) = 1$), it follows that the $R_{r,u,v,w}$ -module $\text{Hom}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{r,u,v,w}), \Phi/\mathcal{O}\right)$ is cyclic by Nakayama's Lemma. The statement for $\mathfrak{T}_{r,u,v}$ is proved in an identical fashion, using Proposition 4.17(iv) instead of Proposition 4.17(iii). \square

Remark 4.19. There exists infinitely many n as in the statement of Corollary 4.18 thanks to [MR04, §4.1].

4.2.2. The lower bound. Let $\bar{n} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$ and $\bar{s} = (r, u, v) \in (\mathbb{Z}_{>0})^3$.

Proposition 4.20. *For $n \in \mathcal{N}_{r,u,v,w}$ (resp., $n \in \mathcal{N}_{r,u,v}$), we have*

$$\begin{aligned} & \text{length}_{\mathcal{O}}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}})\right) - \text{length}_{\mathcal{O}}\left(H_{\mathcal{F}_{\text{can}}(n)^*}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}^*)\right) = \text{length}_{\mathcal{O}}(R_{\bar{n}}) \\ & (\text{resp.,} \\ & \text{length}_{\mathcal{O}}\left(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathfrak{T}_{\bar{s}})\right) - \text{length}_{\mathcal{O}}\left(H_{\mathcal{F}_{\text{can}}(n)^*}^1(\mathbb{Q}, \mathfrak{T}_{\bar{s}}^*)\right) = \text{length}_{\mathcal{O}}(\mathfrak{R}_{\bar{s}}).) \end{aligned}$$

Proof. By [MR04, Corollary 2.3.6] it suffices to verify the assertions of the proposition only when $n = 1$.

Let $\mathbb{T}_{u,v,w}$ be as in §1.1, so that $\mathbb{T}_{u,v,w}$ is a free \mathcal{O} -module of rank uvw . Theorem 4.1.13 of [MR04] (applied with the $\mathcal{O}[[G_{\mathbb{Q}}]]$ -representation $T = \mathbb{T}_{u,v,w}$ and its quotient $\mathbb{T}_{\bar{n}} = \mathbb{T}_{u,v,w}/\varpi^r$) shows that

$$\text{length}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}})) - \text{length}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}^*)) = ruvw \cdot \chi(\mathbb{T}) = \text{length}_{\mathcal{O}}(R_{\bar{n}}),$$

as desired. Similarly, repeating the arguments above for the free \mathcal{O} -module $\mathfrak{T}_{u,v}$ (of rank $uv \cdot \dim_{\mathbb{K}}(\mathcal{R}_0)$), we conclude with the second assertion. \square

Corollary 4.21. *For n as in Proposition 4.20,*

- (i) $\text{length}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}})) \geq \text{length}_{\mathcal{O}}(R_{\bar{n}}),$
- (ii) $\text{length}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathfrak{T}_{\bar{s}})) \geq \text{length}_{\mathcal{O}}(\mathfrak{R}_{\bar{s}}).$

We are now ready to prove Theorem 3.2. Let $\bar{n} = (r, u, v, w)$ and $\bar{s} = (r, u, v)$.

Corollary 4.22. *Let $n \in \mathcal{N}_{\bar{n}}$ (resp., $n \in \mathcal{N}_{\bar{s}}$) be a core vertex for the Selmer structure \mathcal{F}_{can} on the residual representation \bar{T} .*

- (i) *The $R_{\bar{n}}$ -module $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}})$ (resp., the $\mathfrak{R}_{\bar{s}}$ -module $H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{s}})$) is free of rank one.*
- (ii) $H_{\mathcal{F}_{\text{can}}(n)^*}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}^*) = 0$ (resp., $H_{\mathcal{F}_{\text{can}}(n)^*}^1(\mathbb{Q}, \mathfrak{T}_{\bar{s}}^*) = 0$).

Proof. It follows by Corollary 4.18 and Corollary 4.21 that $\text{Hom}(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathbb{T}_{\bar{n}}), \Phi/\mathcal{O})$ (resp., $\text{Hom}(H_{\mathcal{F}_{\text{can}}(n)}^1(\mathbb{Q}, \mathfrak{T}_{\bar{s}}), \Phi/\mathcal{O})$) is a free $R_{\bar{n}}$ -module (resp., a free $\mathfrak{R}_{\bar{s}}$ -module) of rank one. (i) follows from the Gorenstein property of R and \mathfrak{R} , c.f., [Gro67, Prop. 4.9 and 4.10]. The point is that Φ/\mathcal{O} is an injective hull of \mathbb{K} and thus a dualizing module for R and \mathfrak{R} .

(ii) follows from (i) and Proposition 4.20. \square

5. APPLICATIONS

5.1. Example: Elliptic curves and deformations of Kato's Kolyvagin sytem. Suppose E/\mathbb{Q} is an elliptic curve without CM which has split-multiplicative reduction at all primes ℓ dividing its conductor N_E . Let $\bar{T} = E[p]$ be the p -torsion on E . Let

$$\bar{\rho}_E : G_{\mathbb{Q}} \longrightarrow \text{GL}(\bar{T}) = \text{GL}_2(\mathbb{F}_p)$$

be the mod p Galois representation attached to E . Suppose that the universal deformation problem for $\bar{\rho}_E$ is unobstructed; see Example 1.1 above for the content of this assumption. Let R be the universal deformation ring and ρ_E the universal deformation of $\bar{\rho}_E$ and the \mathbb{T} the deformation space on which $G_{\mathbb{Q}}$ acts by ρ_E . Since we assumed that the deformation problem is unobstructed, $R \cong \mathbb{Z}_p[[X_1, X_2, X_3]]$ and \mathbb{T} is free of rank two as an R -module. The Weil-pairing shows that $\chi(\bar{T}) = 1$.

We further suppose that \mathbb{T} satisfies the hypotheses **(H1)**-**(H4)** as well as **(H.Tam)**(i) and **(H.nA)**. Before discussing the applications of Theorem 3.12 to this setting, we first explain the contents of the extra hypotheses **(H.Tam)** and **(H.nA)** in this particular setting.

Proposition 5.1. *Suppose E/\mathbb{Q} is an elliptic curve as above and $\ell | N_E$ is a prime. Assume that*

- p does not divide the Tamagawa number c_{ℓ} at ℓ ,

- p does not divide $\ell - 1$.

Then **(H.Tam)** holds true for $\bar{T} = E[p]$.

Proof. Let $T = T_p(E)$ be the p -adic Tate module of E . Under the running assumptions, there is a non-split exact sequence

$$(5.1) \quad 0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow T \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

of $\mathbb{Z}_p[[G_\ell]]$ -modules. Let $\sigma = \partial(1) \in H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1))$ where $\partial : \mathbb{Z}_p \rightarrow H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1))$ is the connecting homomorphism in the long exact sequence of the G_ℓ -cohomology of the sequence (5.1). Kummer theory gives an isomorphism

$$(5.2) \quad \text{ord}_\ell : H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1)) \xrightarrow{\sim} \mathbb{Q}_\ell^{\times, \wedge} \xrightarrow{\sim} \mathbb{Z}_p.$$

According to [CE56] pp. 290 and 292, $-\sigma$ is the extension class of the sequence (5.1) inside $\text{Ext}_{\mathbb{Z}_p[G_\ell]}^1(\mathbb{Z}_p, \mathbb{Z}_p(1)) = H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1))$. Hence $\text{ord}_\ell(\sigma) \neq 0$ as the sequence (5.1) is non-split and by [Büy11a, Prop. 3.3] it further follows that $\text{ord}_\ell(\sigma) \in \mathbb{Z}_p^\times$ and therefore ∂ is surjective.

We have the following diagram below with exact rows and commutative squares

$$\begin{array}{ccccccc} H^0(\mathbb{Q}_\ell, T) & \longrightarrow & \mathbb{Z}_p & \xrightarrow{\partial} & H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p \\ \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^0(\mathbb{Q}_\ell, \mu_p) \longrightarrow & H^0(\mathbb{Q}_\ell, \bar{T}) \longrightarrow & \mathbb{Z}/p\mathbb{Z} \xrightarrow{\bar{\partial}} & H^1(\mathbb{Q}_\ell, \mu_p) \cong \mathbb{Z}/p\mathbb{Z} \end{array}$$

This shows that the map $\bar{\partial}$ is surjective as well and hence

$$H^0(\mathbb{Q}_\ell, \bar{T}) \cong H^0(\mathbb{Q}_\ell, \mu_p) = 0$$

as we assumed $p \nmid \ell - 1$. □

The assumption **(H.nA)** is to say that p is not *anomalous* for E (in the sense of Mazur [Maz72]). Given an elliptic curve E , Mazur in loc.cit. points out that anomalous primes should be scarce.

Let f be any elliptic newform of weight $\omega \geq 2$ and let

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathcal{O}_f)$$

be the Galois representation attached to f by Deligne with coefficients in the ring of integers \mathcal{O}_f of a finite extension Φ_f of \mathbb{Q}_p . Let T_f be the free \mathcal{O}_f module of rank 2 on which $G_{\mathbb{Q}}$ acts via ρ_f . Let \mathfrak{m}_f denote the maximal ideal of \mathcal{O}_f and let $\bar{\rho}_f$ the residual representation of $\bar{\rho}_f \bmod \mathfrak{m}_f$. Suppose that $\bar{\rho}_f \cong \bar{\rho}$ so that ρ_f is a deformation of $\bar{\rho}$ to the ring \mathcal{O}_f . We thus have a ring homomorphism $\varphi_f : R \rightarrow \mathcal{O}_f$ that induces an isomorphism $T_f \cong \mathbb{T} \otimes_{\varphi_f} \mathcal{O}_f$, and by functoriality a commutative diagram

$$(5.3) \quad \begin{array}{ccc} \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) & \xrightarrow{\varphi_f} & \overline{\mathbf{KS}}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P}) \\ & \searrow & \swarrow \\ & \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) & \end{array}$$

Let $\kappa \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ be any big Kolyvagin system which generates the cyclic R -module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ and let $\varphi_f(\kappa)$ be its image in $\overline{\mathbf{KS}}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P})$. Let $\kappa_f^{\text{Kato}} \in \overline{\mathbf{KS}}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P})$ be the Kolyvagin system obtained from Kato's Euler system constructed in [Kat04] for the modular form f .

Theorem 5.2 (Interpolation). *There is a $\lambda_f \in \mathcal{O}_f$ such that*

$$\lambda_f \cdot \varphi_f(\kappa) = \kappa_f^{\text{Kato}}.$$

Remark 5.3. Theorem 5.2 states that an *improved* version of Kato's Kolyvagin systems (by the factor λ_f) interpolate to give rise to the big Kolyvagin system, rather than Kato's Kolyvagin systems themselves. We call the Kolyvagin system $\varphi_f(\kappa)$ an *improvement* to κ_f^{Kato} as the bounds obtained using the Kolyvagin system $\varphi_f(\kappa)$ improves those obtained using $\kappa_f^{\text{Kato}} = \lambda_f \cdot \varphi_f(\kappa)$ by a factor of λ_f . In particular, when the Kolyvagin system κ_f^{Kato} is itself primitive (in the sense of [MR04, Definition 4.5.5]), then $\lambda_f \in \mathcal{O}_f^\times$.

Proof of Theorem 5.2. Let $\bar{\kappa}$ be the image of κ in $\mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$. By Theorem 3.12, $\bar{\kappa} \neq 0$, so it follows by [MR04, Theorem 5.2.10(ii)] and the commutative diagram (5.3) that $\varphi_f(\kappa)$ also generates the free \mathcal{O}_f -module $\mathbf{KS}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P})$ of rank one. \square

In §5.4 below we will explain how the big Kolyvagin system κ may be used to obtain bounds on certain Selmer groups. Before that, we remark that

- (i) the Kolyvagin system κ_f^{Kato} is related to a special value of L -function attached to f , by the work of Kato [Kat04, §14];
- (ii) the recent works of Emerton [Eme11] and Kisin (unpublished lecture notes) show that the classical modular points are Zariski dense in $\text{Spec}(\mathbb{T})$.

Thanks to these remarks, one hopes that the bounds we shall obtain below in terms κ will be ultimately related to an appropriate p -adic L -function (whose existence for the time being is highly conjectural) in 3-variables. See §5.4.2 for an elaboration of this point.

5.2. Example: Hida's nearly ordinary deformation. Suppose E/\mathbb{Q} , is an elliptic curve without CM and $\bar{\rho}_E$ and $\bar{T} = E[p]$ are defined as in §5.1. Suppose in this section that E is p -ordinary and has split multiplicative reduction at every prime ℓ dividing its conductor N_E . Let f_E be the newform of weight 2 and level N_E associated to E by the work of Wiles and Taylor-Wiles.

Let $\Gamma^w = 1 + p\mathbb{Z}_p$. Identify $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$ by μ_{p-1} via the Teichmüller character ω so that we have

$$\mathbb{Z}_p^\times \cong \Delta \times \Gamma^w.$$

Set $\Lambda^w = \mathbb{Z}_p[[\Gamma^w]]$. Let $\mathfrak{h}^{\text{ord}}$ Hida's universal ordinary Hecke algebra parametrizing Hida family passing through f_E , which is finite flat over Λ^w by [Hid86a, Theorem 1.1]. We will recall some basic properties of $\mathfrak{h}^{\text{ord}}$, for details the reader may consult [Hid86a, Hid86b] and [EPW06, §2] for a survey. The eigenform $f = f_E$ fixed as above corresponds to an *arithmetic specialization*

$$\mathfrak{s}_f : \mathfrak{h}^{\text{ord}} \longrightarrow \mathbb{Z}_p$$

Decompose $\mathfrak{h}^{\text{ord}}$ into a direct sum of its completions at maximal ideals and let $\mathfrak{h}_{\mathfrak{m}}^{\text{ord}}$ be the (unique) summand through which \mathfrak{s}_f factors. The localization of $\mathfrak{h}^{\text{ord}}$ at $\ker(\mathfrak{s}_f)$ is a discrete valuation ring [Nek06, §12.7.5], and hence there is a unique minimal prime $\mathfrak{a} \subset \mathfrak{h}_{\mathfrak{m}}^{\text{ord}}$ such that \mathfrak{s}_f factors through the integral domain

$$(5.4) \quad \mathcal{R} = \mathfrak{h}_{\mathfrak{m}}^{\text{ord}} / \mathfrak{a}.$$

The Λ^w -algebra \mathcal{R} is called the branch of the Hida family on which f_E lives, by duality it corresponds to a family \mathbb{F} of ordinary modular forms. Hida [Hid86b] gives a construction of a big $G_{\mathbb{Q}}$ -representation \mathcal{T} with coefficients in \mathcal{R} and the ring \mathcal{R} is Gorenstein of dimension two (as in [Wil95, TW95]). Let $\Lambda = \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})]]$ be the cyclotomic Iwasawa algebra and set

$\mathfrak{R} = \mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda$ and $\mathfrak{T} = \mathcal{T} \otimes_{\mathbb{Z}_p} \Lambda$, where we allow $G_{\mathbb{Q}}$ act both on \mathcal{T} and Λ . Following Ochiai, we call \mathfrak{T} the universal ordinary deformation of \bar{T} .

Suppose that the hypotheses **(H1)**-**(H4)** as well as **(H.Tam)** and **(H.nA)** hold true. In this case, Theorem 3.12 recovers a theorem of Ochiai [Och05], which he establishes by deforming Kato's Euler system¹ along the ordinary locus of the universal deformation space. For any ordinary eigenform g that lives in the branch \mathcal{R} of the Hida family, let

$$\mathfrak{s}_g : \mathcal{R} \longrightarrow \mathcal{O}_g$$

denote the corresponding arithmetic specialization and $T_g = \mathcal{T} \otimes_{\mathfrak{s}_g} \mathcal{O}_g$ the associated Galois representation, where \mathcal{O}_g is the integers of a finite extension of \mathbb{Q}_p . Let

$$\kappa^{\text{Kato}} \in \overline{\text{KS}}(T_g \otimes \Lambda, \mathcal{F}_{\text{can}}, \mathcal{P})$$

be the Λ -adic Kolyvagin system for to the cyclotomic deformation $T_g \otimes \Lambda$ obtained from Kato's Euler system as in [MR04, §6.2]. The proof of the following is identical to that of Theorem 5.2.

Theorem 5.4. *Let κ be a generator of the free \mathfrak{R} -module $\overline{\text{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ of rank one. Let*

$$\overline{\text{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) \xrightarrow{\mathfrak{s}_g} \overline{\text{KS}}(T_g \otimes \Lambda, \mathcal{F}_{\text{can}}, \mathcal{P})$$

be the map induced from the arithmetic specialization \mathfrak{s}_g above by functoriality. There is a $\lambda_g \in \mathcal{O}_g[[\Gamma]]$ such that

$$\lambda_g \cdot \mathfrak{s}_g(\kappa) = \kappa^{\text{Kato}}.$$

Corollary 5.5. *Let κ be as above. Then its leading term $\kappa_1 \in H^1(\mathbb{Q}, \mathfrak{T})$ is non-zero.*

Proof. For g as above, Kato proves that $\kappa_1^{\text{Kato}} \in H^1(\mathbb{Q}, T_g \otimes \Lambda)$ is non-zero. Corollary follows from Theorem 5.4. \square

Theorem 5.4 may be used along with a standard Kolyvagin system argument to recover the results of [Och05].

5.3. Weak Leopoldt Conjecture for Galois deformations. Let $\mathfrak{R} = \mathcal{R}[[\Gamma]]$ and $\mathfrak{T} = \mathcal{T} \otimes \Lambda$ be as in §1, satisfying the hypotheses **(H1)**-**(H4)**, **(H.Tam)** and **(H.nA)**. Suppose further that $\chi(\bar{T}) = 1$. For any \mathfrak{R} -module M , let $M^\vee = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ denote the Pontruagin dual of M .

Lemma 5.6. *Suppose M is an \mathfrak{R} -module. Assume for a height one prime \wp of \mathcal{R} and an integer N , the quotient $M/(\wp, \gamma - 1 + p^N)M$ is of finite order. Then M is a finitely generated \mathfrak{R} -torsion module.*

Proof. To ease notation, write $\gamma_N = \gamma - 1 + p^N$. Then once can find an integer s so that

$$(5.5) \quad p^s \cdot (M/(\wp, \gamma_N)M) = 0.$$

By Nakayama's Lemma, M is finitely generated as an \mathfrak{R} -module, say by $m_1, \dots, m_r \in M$. It follows from (5.5)

$$p^s m_i = \sum_{j=1}^r a_j^{(i)} m_j,$$

¹Whereas we carry this out in the level of Kolyvagin systems.

where $a_j^{(i)} \in (\wp, \gamma_N)$. Setting $A = [a_j^{(i)}]$ and $B = A - p^s \cdot I_{r \times r}$, we conclude by (5.5) that

$$\begin{aligned} \sum_{j=1}^{i-1} a_j^{(i)} m_j + (a_i^{(i)} - p^s) m_i + \sum_{j=i+1}^r a_j^{(i)} m_j &= 0 \\ \implies (A - p^s \cdot I_{r \times r}) \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} &= B \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = 0 \\ \implies \text{adj}(B) \cdot B \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} &= 0 \implies \det(B) \cdot M = 0. \end{aligned}$$

To conclude with the proof of the lemma, we check that $\det(B) \in \mathfrak{R}$ is non-zero. Observe that

$$\begin{aligned} \det(B) &= \det(A - p^s \cdot I_{r \times r}) \equiv (-1)^r p^{sr} \pmod{(\wp, \gamma_N)} \\ &\not\equiv 0 \pmod{(\wp, \gamma_N)}, \end{aligned}$$

as the ring $\mathfrak{R}/(\wp, \gamma_N) \cong \mathcal{R}/\wp$ is an integral domain of characteristic zero. \square

The goal of this section is to prove Theorems 5.7 and 5.10.

Theorem 5.7. *Suppose the ring \mathcal{R} above is regular. If the \mathfrak{R} -module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is torsion, then there is a Kolyvagin system $\kappa \in \overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ such that:*

- (i) *Its leading term $\kappa_1 \in H^1(\mathbb{Q}, \mathfrak{T})$ is non-vanishing.*
- (ii) *$\kappa_1 \notin \mathfrak{p}H^1(\mathbb{Q}, \mathfrak{T})$ for infinitely many height one primes \mathfrak{p} of \mathfrak{R} .*

Remark 5.8. In the setting of §5.2 above, the ring \mathcal{R} is often regular as explained in [FO11, Lemma 2.7].

Proof. For any ideal I of \mathfrak{R} and any subquotient M of \mathfrak{T} , we have thanks to our running hypothesis **(H3)**, [MR04, 3.5.2] and the proof of [MR04, Lemma 3.5.3] that

$$H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, M^*[I]) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, M^*)[I]$$

and hence also an injection

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, M^*[I]) \hookrightarrow H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, M^*)[I],$$

where $\mathcal{F}_{\text{can}}^*$ on M^* is induced from the Selmer structure \mathcal{F}_{can} on \mathfrak{T}^* by propagation. Passing to Pontryagin duals, we thus obtain a surjection

$$(5.6) \quad H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, M^*)^\vee \otimes \mathfrak{R}/I \twoheadrightarrow H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, M^*[I])^\vee.$$

By the assumption that $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is \mathfrak{R} -torsion, one can choose by [Mat89, Theorem 6.5] a specialization

$$\mathfrak{s}_\wp : \mathcal{R} \longrightarrow S_\wp$$

into the ring of integers S_\wp of a finite extension Φ_\wp of \mathbb{Q}_p , whose kernel \wp is a height one prime $\wp \subset \mathcal{R}$ and satisfies with the following properties:

- S_\wp is integral closure of the integral domain $\mathcal{O}_\wp := \mathcal{R}/\wp$ in $\text{Frac}(\mathcal{O}_\wp) = \Phi_\wp$,

- $\wp \notin \text{Supp}_{\mathfrak{R}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee)$, where \wp here denotes by slight abuse the height one prime which is the kernel of the induced map

$$\mathfrak{R} = \mathcal{R}[[\Gamma]] \xrightarrow{\mathfrak{s}_\wp} S_\wp[[\Gamma]].$$

We denote the induced ring homomorphism $\mathcal{O}_\wp \hookrightarrow S_\wp$ also by \mathfrak{s}_\wp .

For \wp chosen as above, it follows that the module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee / \wp$ is $\mathcal{O}_\wp[[\Gamma]]$ -torsion. By (5.6), this implies that the module

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*[\wp])^\vee \cong H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, (\mathcal{T}/\wp\mathcal{T} \otimes \Lambda)^*)^\vee$$

is $\mathcal{O}_p[[\Gamma]]$ -torsion as well. It is therefore possible (using Hensel's Lemma and [Mat89, Theorem 6.5]) to choose an $N \gg 0$ such that

- $\mathcal{O}_\wp[[\Gamma]]/(\gamma - 1 + p^N) \cong \mathcal{O}_\wp$,
- $\gamma - 1 + p^N \notin \text{Supp}_{\mathcal{O}_\wp[[\Gamma]]}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, (\mathcal{T}/\wp\mathcal{T} \otimes \Lambda)^*)^\vee)$.

For N chosen as above, we therefore have that the module

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, (\mathcal{T}/\wp\mathcal{T} \otimes \Lambda)^*)^\vee / (\gamma - 1 + p^N)$$

is \mathcal{O}_\wp -torsion. Setting $T(\wp, N) := \mathcal{T}/\wp\mathcal{T} \otimes \Lambda / (\gamma - 1 + p^N)$ and applying (5.6) again, we conclude that the module

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, (\mathcal{T}/\wp\mathcal{T} \otimes \Lambda / (\gamma - 1 + p^N))^*)^\vee \cong H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp, N)^*)^\vee$$

is \mathcal{O}_\wp -torsion, hence finite.

When we do not vary N , we write $T(\wp)$ in place of $T(\wp, N)$ to ease notation. Let $T = T(\wp) \otimes_{\mathfrak{s}_\wp} S_\wp$ and define the Selmer structure \mathcal{F}_T by setting

- $\Sigma(\mathcal{F}_T) = \Sigma(\mathcal{F}_{\text{can}}) =: \Sigma$,
- $H_{\mathcal{F}_T}^1(\mathbb{Q}_p, T) = H^1(\mathbb{Q}_p, T)$,
- $H_{\mathcal{F}_T}^1(\mathbb{Q}_\ell, T) = \ker(H^1(\mathbb{Q}_\ell, T) \rightarrow H^1(I_\ell, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p))$, for $\ell \neq p$.

Note that \mathcal{F}_T is exactly what Mazur and Rubin call the *canonical Selmer structure* on T . Let ι denote the injection $T(\wp) \hookrightarrow T$. Then ι induces maps

$$\begin{aligned} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*) &\longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T(\wp)^*) \\ H^1(\mathbb{Q}_\ell, T(\wp)) &\longrightarrow H^1(\mathbb{Q}_\ell, T) \\ H^1(\mathbb{Q}_\ell, T^*) &\longrightarrow H^1(\mathbb{Q}_\ell, T(\wp)^*) \end{aligned}$$

for every prime ℓ . It is easy to see that the image of $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, T(\wp))$ lands in $H_{\mathcal{F}_T}^1(\mathbb{Q}_\ell, T)$ for every ℓ (and by local duality, the image of $H_{\mathcal{F}_T}^1(\mathbb{Q}_\ell, T^*)$ therefore lands in $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, T(\wp)^*)$).

We thence obtain a map

$$(5.7) \quad H_{\mathcal{F}_T}^1(\mathbb{Q}, T^*) \longrightarrow H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp, N)^*)$$

In Lemma 5.9 below we check that the kernel and the cokernel of this map is finite for $N \gg 0$. This shows that $H_{\mathcal{F}_T}^1(\mathbb{Q}, T^*)$ is of finite order for $N \gg 0$, as we have already verified above that $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp, N)^*)$ is finite.

Let $\kappa \in \overline{\text{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ be a generator so that its image $\bar{\kappa} \in \text{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero by Theorem 3.12. Hence, the image $\kappa^{(\wp)}$ of κ in $\overline{\text{KS}}(T, \mathcal{F}_T, \mathcal{P})$ is non-zero as well. Corollary 5.2.13 of [MR04] applies thanks to our running hypotheses and it follows that the $\kappa_1^{(\wp)} \neq 0$ and hence $\kappa_1 \neq 0$, proving (i). Furthermore, the fact that $\kappa_1^{(\wp)} \neq 0$ shows that

$$\kappa_1 \notin \ker(H^1(\mathbb{Q}, \mathfrak{T}) \rightarrow H^1(\mathbb{Q}, \mathcal{T}/\wp\mathcal{T} \otimes \Lambda)) = \wp H^1(\mathbb{Q}, \mathfrak{T})$$

where the final equality is because any height one prime of the regular ring \mathcal{R} (in particular \wp) is principle. This proves (ii). \square

Lemma 5.9. *When the positive integer N that appears in the proof of Theorem 5.7 is sufficiently large, the kernel and the cokernel of the map*

$$H_{\mathcal{F}_T^*}^1(\mathbb{Q}, T^*) \longrightarrow H_{\mathcal{F}_{\text{can}}^*}^1(\mathbb{Q}, T(\wp, N)^*)$$

are both finite.

Proof. We first verify that the kernels and the cokernels of the maps

$$(5.8) \quad H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*) \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T(\wp, N)^*)$$

$$(5.9) \quad H_{\mathcal{F}_T^*}^1(\mathbb{Q}_\ell, T^*) \longrightarrow H_{\mathcal{F}_{\text{can}}^*}^1(\mathbb{Q}_\ell, T(\wp, N)^*)$$

have finite order for every prime ℓ . When we do not vary N , we will denote $T(\wp, N)$ simply by $T(\wp)$.

The kernel of (5.8) lives in $H^0(\mathbb{Q}_\Sigma/\mathbb{Q}, (T/T(\wp))^*)$ and its cokernel in $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, (T/T(\wp))^*)$ which are both finite.

As for the map (5.9) when $\ell = p$, our running hypothesis **(H.nA)** along with the fact that the ideal \wp is principal² (being a height-one prime of the regular ring \mathcal{R}) show that

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_p, T(\wp)) := \text{im} (H^1(\mathbb{Q}_p, \mathfrak{T}) \longrightarrow H^1(\mathbb{Q}_p, T(\wp))) = H^1(\mathbb{Q}_p, T(\wp)),$$

hence we have that

$$H_{\mathcal{F}_T^*}^1(\mathbb{Q}_\ell, T^*) = 0 = H_{\mathcal{F}_{\text{can}}^*}^1(\mathbb{Q}_\ell, T(\wp)^*)$$

so the kernel and cokernel of (5.9) are trivial. It remains to control the kernel and the cokernel of (5.9) when $\ell \neq p$. The kernel of

$$(5.10) \quad H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_\ell, T(\wp)) \longrightarrow H_{\mathcal{F}_T}^1(\mathbb{Q}_\ell, T)$$

is controlled by

$$\ker (H^1(\mathbb{Q}_\ell, T(\wp)) \longrightarrow H^1(\mathbb{Q}_\ell, T)) = \text{im} (H^0(\mathbb{Q}_\ell, T/T(\wp)) \longrightarrow H^1(\mathbb{Q}_\ell, T(\wp)))$$

which is finite.

We finally prove that the cokernel of (5.10) is finite. Consider now the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{ur}}^1(\mathbb{Q}_\ell, T(\wp)) & \longrightarrow & H^1(\mathbb{Q}, T(\wp)) & \longrightarrow & H^1(I_\ell, T(\wp))^{\text{Fr}_\ell=1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_{\text{ur}}^1(\mathbb{Q}_\ell, T) & \longrightarrow & H^1(\mathbb{Q}, T) & \longrightarrow & H^1(I_\ell, T)^{\text{Fr}_\ell=1} \longrightarrow 0 \end{array}$$

The cokernel of the vertical map in the middle is controlled by $H^2(\mathbb{Q}_\ell, T/T(\wp))$ hence it is finite. Also, the kernel of the rightmost is finite for a similar reason. This shows by snake lemma that the cokernel of the leftmost vertical map is also finite. As the index of $H_{\text{ur}}^1(\mathbb{Q}_\ell, T)$ in $H_f^1(\mathbb{Q}_\ell, T) = H_{\mathcal{F}_T}^1(\mathbb{Q}_\ell, T)$ is finite as well, we therefore proved that

$$(5.11) \quad \text{the cokernel of the map } H_{\text{ur}}^1(\mathbb{Q}_\ell, T(\wp)) \longrightarrow H_{\mathcal{F}_T}^1(\mathbb{Q}_\ell, T) \text{ is finite.}$$

Furthermore, it is not hard to see that the Λ -module

$$H_{\text{ur}}^1(\mathbb{Q}_\ell, \mathcal{T}/\wp\mathcal{T} \otimes \Lambda) = H^1(G_\ell/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda)$$

²This is the only point in the proof of Theorems 5.7 and 5.10 where we use that the ring \mathcal{R} is regular in an essential way.

is Λ -torsion. (Note in the equality above we use the fact that I_ℓ acts trivially on Λ .) Choosing the positive integer $N \gg 0$ above so that $\gamma - 1 + p^N$ does not divide the characteristic ideal of this module, we obtain a finite quotient $H^1(G_\ell/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda)/(\gamma - 1 + p^N)$. Since the cohomological dimension of G_ℓ/I_ℓ is one, we have

$$\begin{aligned} H^1(G_\ell/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda)/(\gamma - 1 + p^N) &\xrightarrow{\sim} H^1(G_\ell/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda/(\gamma - 1 + p^N)) \\ &\cong H^1(G_\ell/I_\ell, (\mathcal{T}/\wp\mathcal{T} \otimes \Lambda/(\gamma - 1 + p^N))^{I_\ell}) \\ &\cong H^1(G_\ell/I_\ell, T(\wp, N)^{I_\ell}) = H_{\text{ur}}^1(\mathbb{Q}_\ell, T(\wp, N)), \end{aligned}$$

where $T(\wp, N) = \mathcal{T}/\wp\mathcal{T} \otimes \Lambda/(\gamma - 1 + p^N)$ as above. In particular, the index of $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp, N))$ in $H_{\text{ur}}^1(\mathbb{Q}_\ell, T(\wp, N))$ is finite for $N \gg 0$. This, together with (5.11) shows that the kernel and cokernel of the map (5.10), and by local duality, also the kernel and the cokernel of the map (5.9) are finite for $N \gg 0$.

Using the fact that the kernels and cokernels of the maps (5.8) and (5.9) are both finite the proof of the lemma follows at once. \square

Theorem 5.10. *Suppose the ring \mathcal{R} above is regular and suppose that there is a Kolyvagin system $\kappa \in \overline{\text{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ such that $\kappa_1 \in H^1(\mathbb{Q}, \mathfrak{T})$ is non-torsion. Then the \mathfrak{R} -module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is torsion.*

Proof. As the proof of this Theorem in fact follows from a more general statement due to Ochiai [Och05] (see the proof Theorem 2.6; also §5.4 below), we give a sketch of the proof. We use the notation from the proof of Theorem 5.7.

Since κ_1 is non-torsion, it follows that there is a height one prime \wp of \mathcal{R} as in the proof of Theorem 5.7 and a positive integer N (chosen in way that the conclusion of Lemma 5.9 holds true) such that the image

$$\text{red}_{\wp, N}(\kappa_1) \in H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp, N))$$

of κ_1 is non-zero. By our running hypothesis **(H3)**, the map

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T(\wp)) \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T)$$

is injective. In particular, the image $\kappa_1^{(\wp)}$ of $\text{red}_{\wp, N}(\kappa_1)$ inside $H_{\mathcal{F}_T}^1(\mathbb{Q}, T)$ is non-zero. We therefore conclude for the image $\kappa^{(\wp)} \in \overline{\text{KS}}(T, \mathcal{F}_T, \mathcal{P})$ of κ that its leading term $\kappa_1^{(\wp)} \neq 0$. This shows that $H_{\mathcal{F}_T}^1(\mathbb{Q}, T^*)$ is finite.

By Lemma 5.9, we have a map

$$H_{\mathcal{F}_T}^1(\mathbb{Q}, T^*) \longrightarrow H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp, N)^*)$$

with finite kernel and cokernel. Hence $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T(\wp)^*)$ is finite as well. We conclude by (5.6) that

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee/(\wp, \gamma - 1 + p^N) \cong H_{\mathcal{F}_T}^1(\mathbb{Q}, T^*)^\vee$$

is also finite. It follows from Lemma 5.6 that $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is \mathfrak{R} -torsion, as desired. \square

Remark 5.11. The proof of the Theorem 5.10 shows that the assumption on the leading term of the Kolyvagin system that $\kappa_1 \in H^1(\mathbb{Q}, \mathfrak{T})$ is non-torsion may be weakened to the assumption that $\kappa_1 \notin \wp H^1(\mathbb{Q}, \mathfrak{T})$ for some height one prime \wp of \mathcal{R} . Below we check further that if $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is \mathfrak{R} -torsion, then $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T})$ is a free \mathfrak{R} -module of rank one. In particular, κ_1 is non-torsion and satisfies Theorem 5.7(ii) as well. Hence the assertions of Theorems 5.7 and 5.10 are mutual converses under our running hypotheses.

Remark 5.12. In this remark, we verify the following statement:

- (†) If $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is \mathfrak{R} -torsion then, under the assumptions of Theorem 5.7, the \mathfrak{R} -module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T})$ is free of rank one.

To simplify the arguments, suppose in addition that the ring \mathcal{R} is the power series ring $\mathcal{O}[[X]]$. The general case when \mathcal{R} is a general regular \mathcal{O} -algebra of dimension two may be treated similarly. As above, choose a positive integer $N \gg 0$ so that

- $\mathcal{O}[[X]]/(X + p^N) \cong \mathcal{O}$,
- $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}^*)^\vee/(X + p^N)$ is Λ -torsion.

By setting $T := \mathcal{T}/(X + p^N)\mathcal{T}$, we conclude using (5.6) that the module $H^1(\mathbb{Q}, (T \otimes \Lambda)^*)^\vee$ is Λ -torsion. Similarly, choose a positive integer $M \gg 0$ such that

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, (T \otimes \Lambda)^*)^\vee/(\gamma - 1 + p^M) \cong H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \dot{T})^\vee$$

is finite. Here, \dot{T} is the free \mathcal{O} -module $T \otimes \Lambda/(\gamma - 1 + p^M)$. By [MR04, Corollary 5.2.6], it follows that $\text{rank}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \dot{T})) = \chi(\bar{T}) = 1$. Furthermore, the \mathcal{O} -module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \dot{T})$ is torsion-free as since we assume **(H3)**, hence we conclude that $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \dot{T})$ is a free \mathcal{O} -module of rank one.

Set $X_1 = X + p^N$ and $X_2 = \gamma - 1 + p^M$ for M, N as above and define $\mathfrak{R}_{u,v} = \mathfrak{R}/(X_1^u, X_2^v)$, $\mathfrak{R}_{r,u,v} = \mathfrak{R}/(\varpi^r, X_1^u, X_2^v)$, $\mathfrak{T}_{u,v} = \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{u,v}$ and $\mathfrak{T}_{r,u,v} = \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{r,u,v}$. Note that $\mathfrak{T}_{1,1} = \dot{T}$. As $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v}) = \varprojlim_r H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{r,u,v})$, it follows by the proof of Prop. 4.17 that

$$H^1(\mathbb{Q}, \mathfrak{T}_{1,1}) \xrightarrow{\sim} H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v})[X_1^{u-1}, X_2^{v-1}].$$

This shows that the module

$$\begin{aligned} \text{Hom}_{\mathfrak{R}_{u,v}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v}), \mathfrak{R}_{u,v})/(X_1^{u-1}, X_2^{v-1}) &\cong \text{Hom}_{\mathfrak{R}_{u,v}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v})[X_1^{u-1}, X_2^{v-1}], \mathfrak{R}_{u,v}) \\ &\cong \text{Hom}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{1,1}), \mathcal{O}), \end{aligned}$$

is cyclic, hence by Nakayama's Lemma (along with the fact that the \mathcal{O} -module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{1,1}) = H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \dot{T})$ is free of rank one) the module $\text{Hom}_{\mathfrak{R}_{u,v}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v}), \mathfrak{R}_{u,v})$ is cyclic as well.

On the other hand, **(H3)** shows that the module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v})$ is \mathcal{O} -torsion free and the proof of Prop. 4.20 shows $\text{rank}_{\mathcal{O}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v})) \geq uv$. This shows that the cyclic $\mathfrak{R}_{u,v}$ -module $\text{Hom}_{\mathfrak{R}_{u,v}}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v}), \mathfrak{R}_{u,v})$ is indeed free of rank one, hence the module $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathfrak{T}_{u,v})$ itself is free of rank one as an $\mathfrak{R}_{u,v}$ -module. Passing to limit we conclude with the proof of the assertion (†).

5.4. Bounding the Selmer group. Let $R = \mathcal{O}[[X_1, X_2, X_3]]$ be as above and suppose M is a finitely generated R -module. Since R is regular, localizations of R at height one primes \mathfrak{p} are discrete valuation rings. If M is torsion, define

$$\text{char}(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{\text{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})}.$$

If M is not torsion, we set $\text{char}(M) = 0$.

The following theorem may be proved (under the running assumptions of §5.3) following the arguments of [Och05], see particularly the proof of Theorem 2.4 in loc.cit.

Theorem 5.13. $\text{char}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathbb{T}^*)^\vee) \mid \text{char}(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, \mathbb{T})/R \cdot \kappa_1).$

5.4.1. *A conjectural example.* Let $d := \text{rank}_R(\mathbb{T})$.

Lemma 5.14. *Under the hypotheses (H.nA), the R -module $H^1(\mathbb{Q}_p, \mathbb{T})$ is free of rank d .*

Proof. The arguments of [Büy09, Appendix] prove under the running assumption that the $R_{u,v,w}$ -module $H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w})$ is free of rank d . The proof of the lemma follows passing to inverse limit. \square

Recall the definition of a pseudo-geometric specialization given in Definition 1.5.

Conjecture 1. *Suppose (H.nA) holds and assume that $\chi(\bar{T}) = 1$. There is an R -submodule (the Bloch-Kato submodule) $H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}_p, \mathbb{T}) \subset H^1(\mathbb{Q}_p, \mathbb{T})$ with the following properties:*

- (i) (Interpolation) *For $\varphi : R \rightarrow \mathfrak{D}$ a pseudo-geometric specialization into a finite, flat, normal extension \mathfrak{D} of \mathbb{Z}_p and $T = \mathbb{T} \otimes_{\varphi} \mathfrak{D}$, $V = T \otimes \mathbb{Q}_p$, the Bloch-Kato submodule $H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}_p, \mathbb{T})$ maps into the Bloch-Kato subgroup*

$$H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}_p, T) := \ker (H^1(\mathbb{Q}_p, T) \rightarrow H^1(\mathbb{Q}_p, V \otimes B_{\text{cris}}))$$

with finite cokernel,

- (ii) *Suppose in addition that the following ‘Pančiškin condition’ holds:*

$$(5.12) \quad H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}_p, \mathbb{T}) \text{ is a free } R\text{-module of rank } d - 1.$$

$$\text{Then, } H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}, \mathbb{T}) := \ker \left(H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathbb{T})}{H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}_p, \mathbb{T})} \right) = 0.$$

We assume until the end that \mathbb{T} satisfies the Pančiškin condition (5.12) and that Conjecture 1 holds true. Set

$$H^1_s(\mathbb{Q}_p, \mathbb{T}) = H^1(\mathbb{Q}_p, \mathbb{T}) / H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}_p, \mathbb{T}).$$

(ii) along with global duality implies that there is an exact sequence of R -modules

$$0 \rightarrow H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T}) / R \cdot c \xrightarrow{\text{loc}_p^s} H^1_s(\mathbb{Q}_p, \mathbb{T}) / R \cdot \text{loc}_p^s(c) \rightarrow H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}, \mathbb{T}^*)^{\vee} \rightarrow H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^{\vee} \rightarrow 0$$

for any class $c \in H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T})$. In particular, if $0 \neq \kappa \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ and $\kappa_1 \in H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T})$, we have

$$(5.13) \quad 0 \rightarrow H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T}) / R \cdot \kappa_1 \xrightarrow{\text{loc}_p^s} H^1_s(\mathbb{Q}_p, \mathbb{T}) / R \cdot \text{loc}_p^s(\kappa_1) \rightarrow H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}, \mathbb{T}^*)^{\vee} \rightarrow H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^{\vee} \rightarrow 0$$

Theorem 5.13 shows that

Corollary 5.15. $\text{char} \left(H^1_{\mathcal{F}_{\text{BK}}}(\mathbb{Q}, \mathbb{T}^*)^{\vee} \right) \mid \text{char} \left(H^1_s(\mathbb{Q}_p, \mathbb{T}) / R \cdot \text{loc}_p^s(\kappa_1) \right).$

5.4.2. *Elliptic curves revisited.* Suppose E/\mathbb{Q} is an elliptic curve. Let $\bar{T} = E[p]$ and \mathbb{T} be as in §5.1, satisfying the hypotheses (H1)-(H4), (H.nOb), (H.nA) and (H.Tam)(i). Then $d = 2$. We assume that \mathbb{T} satisfies (5.12) and Conjecture 1 holds true.

Definition 5.16. For any big Kolyvagin system κ that generates the module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$, the principal ideal

$$\mathcal{L}(\kappa) = \text{char} \left(H^1_s(\mathbb{Q}_p, \mathbb{T}) / R \cdot \text{loc}_p^s(\kappa_1) \right) \subset R$$

is called the *Kolyvagin constructed p -adic L -function*.

This should be thought of as a generalization of Perrin-Riou’s [PR95] module of algebraic p -adic L -function, whose definition she gives for the cyclotomic deformation of a motive.

Let f be an elliptic modular form of weight $\omega \geq 2$ and T_f be the rank-two \mathcal{O}_f -module attached to f by Deligne, as in §5.1. Then by the universality of R , there is a ring homomorphism $\varphi_f : R \rightarrow \mathcal{O}_f$ that induces $\mathbb{T} \otimes_{\varphi_f} \mathcal{O}_f \xrightarrow{\sim} T_f$, which yields by Conjecture 1(i)

$$(5.14) \quad \varphi_f : H_s^1(\mathbb{Q}_p, \mathbb{T}) \longrightarrow H_s^1(\mathbb{Q}_p, T_f)$$

and by Theorem 5.2

$$(5.15) \quad \lambda_f \cdot \varphi_f(\kappa) = \kappa^{\text{Kato}, (f)},$$

where $\lambda_f \in \mathcal{O}_f$ and $\kappa^{\text{Kato}, (f)}$ is Kato's Kolyvagin system for T_f .

Proposition 5.17. *Suppose E above has good ordinary reduction at p and the R -module $H_s^1(\mathbb{Q}_p, \mathbb{T})$ is torsion-free. Then for $\kappa \neq 0$ as above, $\mathcal{L}(\kappa) \neq 0$.*

Proof. For any f that is as above and which is p -ordinary, we have a ring homomorphism

$$\varphi_f^\Lambda : R \longrightarrow \mathcal{O}_f[[\Gamma]]$$

that induces a map

$$\overline{\text{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) \xrightarrow{\varphi_f^\Lambda} \overline{\text{KS}}(T_f \otimes \Lambda, \mathcal{F}_{\text{can}}, \mathcal{P})$$

for which one verifies that

$$(5.16) \quad \varphi_f^\Lambda(\kappa) \neq 0$$

as in the proof Theorem 5.2. On the other hand, Kato [Kat04] proves that the leading term $\kappa_1^{\text{Kato}, (f), \Lambda} \in H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T_f \otimes \Lambda)$ of his Kolyvagin system $\kappa^{\text{Kato}, (f), \Lambda}$ is non-zero. This shows, using [Büy11b, Theorem 3.23] and (5.16), that $\kappa_1 \neq 0$. Thanks to our assumption that $H_s^1(\mathbb{Q}_p, \mathbb{T})$ is R -torsion-free and (5.12), along with the left-most injection in (5.13), it follows that $\text{loc}_p^s(\kappa_1) \neq 0$ and further that the quotient $H_s^1(\mathbb{Q}_p, \mathbb{T})/R \cdot \text{loc}_p^s(\kappa_1)$ is R -torsion. \square

Theorem 5.18. *Suppose that the hypotheses of Proposition 5.17 hold true. Assume further that the R -module $H_s^1(\mathbb{Q}_p, \mathbb{T})$ is free. Then*

$$(5.17) \quad [\mathcal{O}_f : \varphi_f(\mathcal{L}(\kappa))] = \lambda_f^{-1} \cdot \# \left(\frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s(\kappa_1^{\text{Kato}, (f)})} \right).$$

Proof. Observe that $\mathcal{L}(\kappa)$ may be described as the initial fitting ideal $\text{Fitt}_R^0 \left(\frac{H_s^1(\mathbb{Q}_p, \mathbb{T})}{R \cdot \text{loc}_p^s(\kappa_1)} \right)$, and $\varphi_f(\mathcal{L}(\kappa))$ as $\text{Fitt}_{\mathcal{O}_f}^0 \left(\frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s(\varphi_f(\kappa)_1)} \right)$. It follows by (5.15) that

$$\begin{aligned} [\mathcal{O}_f : \varphi_f(\mathcal{L}(\kappa))] &= \lambda_f^{-1} \left[\mathcal{O}_f : \text{Fitt}_{\mathcal{O}_f}^0 \left(\frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s(\varphi_f(\kappa)_1)} \right) \right] \\ &= \lambda_f^{-1} \left(\# \frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s(\kappa_1^{\text{Kato}, (f)})} \right). \end{aligned}$$

\square

A few remarks are in order. Kato has related the values $\# \left(\frac{H_s^1(\mathbb{Q}_p, T_f)}{\mathcal{O}_f \cdot \text{loc}_p^s \left(\kappa_1^{\text{Kato}, (f)} \right)} \right)$ to the value of the L -function attached to f at the central critical point. The statement of Theorem 5.18 therefore suggests that these values should interpolate, as the classes $\lambda_f^{-1} \kappa_1^{\text{Kato}, (f)}$ do interpolate to κ_1 . This hints at the existence of a very general p -adic L -function. Furthermore, we note that the points $\{\varphi_f : \mathbb{T} \rightarrow \mathcal{O}_f : f \text{ is a classical modular form}\}$ are Zariski dense in $\text{Spec}(\mathbb{T})$ under mild hypotheses, thanks to the recent results of Emerton and Kisin. Thus the assertion (5.17) should characterize the ideal $\mathcal{L}(\kappa)$ of the Kolyvagin constructed p -adic L -function.

Acknowledgements. The author wishes to thank Jan Nekovář and Tadashi Ochiai for very useful conversations on various technical points; to Shu Sasaki for his comments on an earlier version of this article and for letting the author know of Kisin's recent $R = T$ theorem. He is indebted to Barry Mazur for encouraging the author to extend his thesis work to this generality. Much of the work in this article was carried out during the two summers when the author was hosted by Pontificia Universidad Católica de Chile, for whose hospitality he is grateful. He acknowledges a Marie Curie grant FP7-230668 and a TÜBİTAK grant 109T662.

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