On the main conjectures for CM fields and Rubin-Stark elements

KÂZIM BÜYÜKBODUK

ABSTRACT. Making use of the conjectural Rubin-Stark elements defined over abelian extensions of a given CM field F, we construct an Euler system of rank g (in the sense of Perrin-Riou, where $2g = [F : \mathbb{Q}]$) for Hecke characters of F. We use this Euler system of rank g to reduce the main conjectures of Iwasawa theory to a local statement about Rubin-Stark elements, which is a natural extension of a result of Yager that relates elliptic units to Katz's two-variable p-adic L-function. We are able to prove this statement in certain cases, thanks to the recent of works of Hida and Hsieh. These results have applications in the arithmetic of abelian varieties which have CM by F.

CONTENTS

1. INTRODUCTION

The main objective of this paper is to study Iwasawa's main conjecture for CM fields (see [HT94, Page 90] for its precise statement). When the CM field in question is an imaginary quadratic number field, the main conjectures were proved by Rubin [Rub91] using the elliptic unit Euler system. For more general CM fields, this has been tackled by Hida and Tilouine [HT94]

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and Hida [Hid09] along the anticyclotmic tower, by Mainardi [Mai08] along the cyclotomic tower and by Hsieh [Hsi12] in more general context. All these works relied on the Eisenstein ideal method. In this paper, we approach this problem via the conjectural Rubin-Stark elements, by means of extending the methods of [Büy10] so as to refine the rank-r Euler-Kolyvagin system machinery developed therein. As a result, we prove Theorem A below, which reduces the main conjectures to Conjecture 2 below, a purely local statement. Conjecture 2 asserts that the collection of Rubin-Stark elements along the maximal $\mathbb{Z}_p$-power extension of a CM field recovers the Katz p-adic L-function. This is a natural and a far reaching generalization of a theorem of Yager [Yag82, Theorem 1] which relates the tower of elliptic units along the $\mathbb{Z}_p^2$-extension of an imaginary quadratic field to the two-variable p-adic L-function of Katz. Under certain technical hypotheses, a recent result of Hsieh [Hsi12] on the CM main conjectures allows us to prove Conjecture 2, and thereby extend Yager's result to a general CM field. This result is stated as Theorem B below.

Our approach sheds light on the CM main conjectures beyond cases covered by [HT94, Hid09, Mai08, Hsi12]. To give an example, suppose $\mathcal{E}$ is an elliptic curve defined over a totally real field K which has CM by the imaginary quadratic field M and set $F = MK$. All prior results concerning the main conjectures for $\mathcal{E}$ require the assumption that the sign $W(\mathcal{E}/K)$ of the functional equation of the Hasse-Weil L-function of $\mathcal{E}_{/K}$ is $+1$. Our Theorem A allows us to go beyond that (of course, it also proves less than the main conjectures, too). Let M_∞^- denote the anticyclotomic $\mathbb{Z}_p$-extension of M. When $W(\mathcal{E}/K) = -1$, the Iwasawa theory of $\mathcal{E}$ along $F_\infty^- := FM_\infty^-$ is much different from the cyclotomic Iwasawa theory. The relevant Selmer group in this case is not expected to be torsion and the methods of [HT94, Hid09, Mai08, Hsi12] does not seem adequate to address this case. When $K = \mathbb{Q}$, this has been studied by Agboola and Howard in [AH06] using the elliptic unit Euler system; see also [Arn07] for a generalization of their results to higher weight forms, as well as [Arn10] for the treatment of an analogous problem in the non-CM case, this time making use of Kato-Ochiai Euler system. The techniques we develop in this paper will be used in our forthcoming work [Büy12b] to study the Iwasawa theory of $\mathcal{E}_{/K}$ along F_∞^-, when K is a general totally real field and when the sign of the functional equation is *not* necessarily $+1$.

As in [HT94, Hid09, Mai08, Hsi12], we work in this paper under a certain p-ordinarity condition, labelled as (**H.pOrd**) below. It seems very likely that the approach via the conjectural Rubin-Stark elements which we develop in this paper could be adopted to the supersingular setting as well, at least in a large number of cases of interest, e.g., when the prime p is assumed to split completely in the totally real field K. This certainly falls away from the scope of the previous works on the CM main conjectures which are alluded to above. We will address this matter in a future paper as well.

Before giving a further account of our results, we set our notation that will be in effect throughout this paper. Let F be a CM field and K its maximal totally real subfield that has degree g over $\mathbb{Q}$. Fix a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ lifting the generator of $\mathrm{Gal}(F/K)$. Fix forever an odd prime p unramified in $F/\mathbb{Q}$ and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Definition 1.1. *A CM-type* Σ is a collection of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ such that $\Sigma \cup \Sigma^c$ is the full set of embeddings of F into $\overline{\mathbb{Q}}$ and $\Sigma \cap \Sigma^c = \emptyset$. Here the complex conjugation σ^c of an embedding σ is defined so that $\sigma^c(a) = \sigma(a^c)$ for $a \in F$.

Suppose that the CM-type Σ satisfies the following hypothesis:
(H.pOrd) The embeddings $\Sigma_p := \{\iota_p \circ \sigma\}_{\sigma\in\Sigma}$ induce exactly half of the places of F over p.

Such a CM-type is called p-ordinary. We identify Σ_p with the associated subset of primes $\{\wp_1, \cdots, \wp_s\}$ of F above p. Setting $\Sigma_p^c = \{\wp_1^c, \cdots \wp_s^c\}$, we see that the disjoint union $\Sigma_p \sqcup \Sigma_p^c$ is the set of all primes of F above p. As explained in [Hid09], there then exists an abelian variety with CM by F having good ordinary reduction at p, and its CM-type is Σ.

Let F_∞ be the maximal $\mathbb{Z}_p$-power extension of F and let $\Gamma = \mathrm{Gal}(F_\infty/F)$. Then $\Gamma \cong \mathbb{Z}_p^{r_2+1+\delta}$, where δ is the Leopoldt defect. Fix a finite extension $\mathfrak{F}$ of $\mathbb{Q}_p$ and denote its ring of integers by $\mathfrak{O}$. Let $\varpi \in \mathfrak{O}$ be a fixed uniformizing element. We let $\mathcal{W}$ be the valuation ring of $\widehat{\overline{\mathbb{Q}}}_p$, the completion of $\overline{\mathbb{Q}}_p$. Set $\Lambda = \mathfrak{O}[[\Gamma]]$ to be the Iwasawa algebra, which is isomorphic to the formal power series ring in $r_2+1+\delta$ variables with coefficients in $\mathfrak{O}$ and define $\Lambda_{\mathcal{W}} := \mathcal{W}[[\Gamma]]$.

Definition 1.2. For a torsion Λ-module $\mathfrak{X}$, define its characteristic ideal

$$\mathrm{char}(\mathfrak{X}) = \prod_{\mathfrak{P}} \mathfrak{P}^{\mathrm{length}(\mathfrak{X}_{\mathfrak{P}})},$$

where the product is over height-one prime of Λ.

Since Λ is regular, note that the ideal $\mathrm{char}(\mathfrak{X})$ is principal. Let $\chi : G_F \to \mathfrak{O}^\times$ be a Dirichlet character of order prime to p and let L_χ be the finite extension of F that χ cuts out. We define M_∞ to be the maximal abelian pro-p extension of $L_\infty = L_\chi F_\infty$ which is unramified outside Σ_p. Set $\mathfrak{X}_\infty = \mathrm{Gal}(M_\infty/L_\infty)^\chi$, which is a finitely generated torsion Λ-module (see [HT94, §1.2]). Then the main conjecture in this setting asserts that:

Conjecture 1 (Main conjecture)**.** *The characteristic ideal of $\mathfrak{X}_\infty \otimes_{\mathfrak{O}} \mathcal{W}$ is generated by the Katz p-adic L-function $\mathcal{L}_\chi^\Sigma \in \mathcal{W}[[\Gamma]]$.*

When $K = \mathbb{Q}$ and F is a quadratic imaginary quadratic number field, Rubin [Rub91] proved this conjecture using the Euler system of elliptic units. Rubin first bounds the characteristic ideal of $\mathfrak{X}_\infty$ in terms of elliptic units using the Euler system machinery, then he uses Yager's theorem [Yag82, Theorem 1] to relate his bound to Katz's two-variable p-adic L-function that was constructed in [Kat76].

In this paper, we generalize Rubin's techniques to general CM-fields, using the conjectural Rubin-Stark elements (see [Rub96] for their conjectural description) in place of elliptic units and extending the general machinery of "Euler-Kolyvagin systems of rank g" set up in [Büy10] to apply in this setting. We prove the following theorem, generalizing [Rub91, Theorems 9.3 and10.6]. For a finite extension $\mathcal{M}$ of L_χ, let Σ_M be the places of of $\mathcal{M}$ that lie above those in Σ and let $U_{\mathcal{M},\mathfrak{p}}$ denote the p-adic completion of the local units of $\mathcal{M}$ at $\mathfrak{p}$ and $U_{\mathcal{M}} = \oplus_{\mathfrak{p}\in\Sigma_{\mathcal{M}}} U_{\mathcal{M},\mathfrak{p}}$ be the semi-local units and $U_{\mathcal{M}}^\chi$ its χ-isotypic part. Set $\mathfrak{U}_\infty^\chi = \varprojlim U_{\mathcal{M}}^\chi$ and $\mathrm{loc}_p^{+,\otimes g}(\varepsilon_{F_\infty}^\chi) \in \wedge^g \mathfrak{U}_\infty^\chi$ be the image of the tower of Rubin-Stark elements inside $\wedge^g \mathfrak{U}_\infty^\chi$; see §3 and §4.2 for a precise definition of the element $\mathrm{loc}_p^{+,\otimes g}(\varepsilon_{F_\infty}^\chi)$.

Theorem A. (See Theorem 4.4 below). *Assume that*

(1) $\chi(\wp) \neq 1$ *for any prime $\wp$ of F above p and $\chi \neq \omega$ (where ω is the p-adic Teichmüller character),*
(2) *Leopolt's conjecture holds for L_χ,*
(3) *Rubin-Stark conjecture is true for all abelian extensions $\mathcal{K}/F$ chosen from the collection $\mathfrak{E}_0$, which is defined as in §1.1 below.*

Then,

$$\mathrm{char}(\mathfrak{X}_\infty) = \mathrm{char}(\wedge^g \mathfrak{U}_\infty^\chi / \Lambda \cdot \mathrm{loc}_p^{+,\otimes g}(\varepsilon_{F_\infty}^\chi)).$$

Here the exterior product is calculated in the category of Λ-modules. See Remark 3.7 for a comment regarding the assumption (3) in the statement of the theorem above. The following conjecture we propose is a natural extension of Yager's theorem [Yag82, Theorem 1] to our setting:

Conjecture 2. *The ideal* char $\left(\wedge^g \mathfrak{U}_\infty^\chi/\Lambda \cdot \mathrm{loc}_p^{+,\otimes g}(\varepsilon_{F_\infty}^\chi)\right)\Lambda_{\mathcal{W}}$ *is generated by* $\mathcal{L}_\chi^\Sigma$.

In view of Theorem A, this conjecture is equivalent to the main conjecture (namely, Conjecture 1 above). In particular, the recent work of Hsieh [Hsi12] gives a partial result towards the truth of Conjecture 2:

Theorem B. *Assume that the hypotheses of Theorem A holds and suppose in addition that:*

1. $p > 5$ *is prime to the minus part of the class number of* F*, to the order of* χ *and is unramified in* $K/\mathbb{Q}$.
2. χ *is* anticyclotomic *in the sense that* $\chi(c\delta c^{-1}) = \chi(\delta)^{-1}$ *for* $\delta \in \Delta$ *and* $c \in G_K$ *that induces the generator of* $\mathrm{Gal}(F/K)$.
3. χ *is unramified at all places above* p.
4. *The restriction of* χ *to* $G_{F(\sqrt{p^*})}$ *(where* $p^* = (-1)^{\frac{p-1}{2}}p$*) is non-trivial.*

Then Conjecture 2 is true.

Assuming the truth of Conjecture 2, the following result on the arithmetic of CM abelian varieties is a consequence of Theorem A; compare to [Rub91, Theorem 11.4]. Let A be an abelian variety defined over F, that has CM by the ring of integers $\mathcal{O}_F$ of F. For a fixed place $\varepsilon \in \Sigma$, let ψ_ε be the associated (archimedean) Hecke character and let $\wp$ be a prime of F above p (whose choice depends in a precise way on the choice of the archimedean avatar ψ_ε of the associated Grössencharacter, see §4.4 below for precise definitions).

Theorem C (Theorem 4.10)**.** *Assume that the hypotheses of Theorem A as well as Conjecture 2 hold true. If* $L(\psi_\varepsilon, 0)$ *is nonzero, then* $A(F)$ *is finite and* $\text{Ш}_{A/F}[\wp^\infty]$ *is finite for sufficiently large primes* p.

See §4.4 below for a detailed discussion; also [Hsi12, Corollary 1] for a result along these lines on the arithmetic of CM elliptic curves, under different set of assumptions and proved using completely different techniques. In a future work, we hope to prove this result without assuming the truth of Conjecture 2, and relying on Theorem A and the refinement of the Kolyvagin System machinery that is used to prove this theorem.

1.1. Notation and Hypotheses.

Given a group G, let $\boldsymbol{\mu}(G)$ denote the torsion subgroup of G. For the local field $\mathfrak{F}$ introduced above, assume that

$$p \text{ does not divide } |\boldsymbol{\mu}(\mathfrak{F}^\times)|. \tag{1.1}$$

Let $\chi : G \to \mathcal{O}^\times$ be any continuous character. For an abelian group A on which G acts continuously, let $\hat{A} : \varprojlim_n A/A^{p^n}$ be the p-adic completion of A and

$$A^\chi = \{a \in \hat{A} \otimes_{\mathbb{Z}_p} \mathcal{O} : ga = \chi(g)a \text{ for all } g \in G\}.$$

For a field k, fix a separable closure $\bar{k}$ of k and let $G_k = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group of k. When k is a global field, $\mathbb{A}_k$ denotes the adèle ring of k and $\mathbb{A}_k^\times$ the group of idéles.

Denote by F^{cyc} the cyclotomic $\mathbb{Z}_p$-extension of F. Set $\Gamma^{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$ and $\Lambda^{\mathrm{cyc}} = \mathfrak{O}[[\Gamma^{\mathrm{cyc}}]]$. Let $\chi_{\mathrm{cyc}} : G_F \to \mathbb{Z}_p^\times$ denote the cyclotomic character giving the action of G_F on the

p-power roots of unity $\boldsymbol{\mu}_{p^\infty}$ and let $\langle\chi_{\text{cyc}}\rangle := \chi_{\text{cyc}}\big|_{\Gamma^{\text{cyc}}} : \Gamma^{\text{cyc}} \to \mathbb{Z}_p^\times$. Then $\omega = \langle\chi_{\text{cyc}}\rangle^{-1}\chi_{\text{cyc}}$ is the Teichmüller character, giving the action of G_F on $\boldsymbol{\mu}_p$.

Suppose $\psi : G_F \twoheadrightarrow \mathfrak{O}^\times$ is a continuous (p-adic) Hecke character with $\text{im}(\psi) = \mathfrak{O}^\times$ and which is Hodge-Tate. Write $\mathfrak{O}^\times = \boldsymbol{\mu}(\mathfrak{F}^\times) \times U^{(1)}$ and let $\langle\psi\rangle : G_F \twoheadrightarrow U^{(1)}$ be the map ψ followed by the projection $\mathfrak{O}^\times \twoheadrightarrow U^{(1)}$. Set

$$\omega_\psi = \psi \cdot \langle\psi\rangle^{-1} : G_F \twoheadrightarrow \boldsymbol{\mu}(\mathfrak{F}^\times) \hookrightarrow \mathfrak{O}^\times$$

and assume that

$$\omega_\psi \text{ is not the trivial character.} \tag{1.2}$$

Let $\psi^* = \psi^{-1} \otimes \chi_{\text{cyc}}$ and $T = \mathfrak{O}(\psi^*)$ be the free $\mathfrak{O}$ module of rank one on which G_F acts via the Hecke character ψ^*. Given a Hecke character ψ as above, let F_ψ/F be the $\mathbb{Z}_p^{[\mathfrak{F}:\mathbb{Q}_p]}$-extension that $\langle\psi\rangle$ factors through. Write $\Gamma_\psi = \text{Gal}(F_\psi/F)$ and $\Lambda_\psi = \mathbb{Z}_p[[\Gamma_\psi]]$.

Let $\chi : G_F \to \mathfrak{O}^\times$ be any Dirichlet character (whose order is necessarily prime to p) which has the property that

$$\chi(\wp) \neq 1 \quad \text{for any prime } \wp \text{ of } F \text{ above } p. \tag{1.3}$$

and that

$$\chi \neq \omega. \tag{1.4}$$

Define the G_F-representation $T_\chi = \mathfrak{O}(1) \otimes \chi^{-1}$. Let L_χ be the finite extension of F cut by the character χ and set $\Delta = \text{Gal}(L_\chi/F)$; note that $p \nmid |\Delta|$. Although for applications of our results to a CM abelian variety A, we will only need to study the case $\chi = \omega_\psi$ where ψ is the p-adic Hecke character attached to A, we still chose to state some of our results in greater generality.

Let $\mathcal{R}$ be the set of primes of F that does not contain any prime above p nor any prime at which χ is ramified. Define $\mathcal{N}(\mathcal{R})$ to be the square free products of primes chosen from $\mathcal{R}$. For $\ell \in \mathcal{R}$, let $F(\ell)$ be the maximal p-extension inside the ray class field of F modulo ℓ and for $\eta = \ell_1 \cdots \ell_s \in \mathcal{N}(\mathcal{R})$, set $F(\eta) = F(\ell_1) \cdots F(\ell_s)$. We write $L(\eta) = L \cdot F(\eta)$ for the composite field. We define the collections of finite abelian extensions of F (resp., of L)

$$\mathfrak{E} = \{M \cdot F(\eta) : \eta \in \mathcal{N}(\mathcal{R}); M \subset F_\infty \text{ is a finite extension of } F\},$$

$$\mathfrak{E}_0 = \{M \cdot L(\eta) : \eta \in \mathcal{N}(\mathcal{R}); M \subset F_\infty \text{ is a finite extension of } F\},$$

Let $\mathfrak{K}_0 = \varinjlim_{N \in \mathfrak{E}_0} N$ and $\mathfrak{K} = \varinjlim_{N \in \mathfrak{E}} N$. Set $\mathfrak{G}(\mathfrak{X}) = \text{Gal}(\mathfrak{X}/F)$ and $\Lambda_{\mathfrak{X}} = \mathfrak{O}[[\mathfrak{G}(\mathfrak{X})]]$ for $\mathfrak{X} = \mathfrak{K}_0$ and $\mathfrak{K}$.

For any non-archimedean prime λ of F, fix a decomposition group $\mathcal{D}_\lambda$ and the inertia subgroup $\mathcal{I}_\lambda \subset \mathcal{D}_\lambda$. Let $(-)^\vee = \text{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ denote Pontryagin duality functor.

2. Selmer structures and Selmer groups

2.1. **Semi-local Preparation.** Let $M = M_0 \cdot F(\eta)$ be a member of the collection $\mathfrak{E}$, where M_0 is a finite subextension of F_∞/F. Set $\Delta_M = \text{Gal}(M/F)$, $\delta_M = |\Delta_M|$ and $\Lambda_M = \mathfrak{O}[\Delta_M]$.

Let X be any $\mathfrak{O}[[G_F]]$-module which is free of rank d as an $\mathfrak{O}$-module. Suppose in addition that X satisfies the following hypothesis:

(H.p1) $H^2(F_\wp, X) = 0 = H^2\left(F_\wp, \text{Hom}_{\mathfrak{O}}(X, \mathfrak{O}(1))\right)$, for any prime $\wp$ of F above p.

Lemma 2.1. *Suppose X is above. Let $M \in \mathfrak{E}$ be an extension of F and let $\mathfrak{P}$ be a prime of M lying above p. Then*

$$H^2(M_{\mathfrak{P}}, X) = 0 = H^2\left(M_{\mathfrak{P}}, \mathrm{Hom}_{\mathfrak{O}}(X, \mathfrak{O}(1))\right).$$

Proof. Let $\wp$ be the prime of F lying below $\mathfrak{P}$ and set $D_{\mathfrak{P}} = \mathrm{Gal}(M_{\mathfrak{P}}/F_{\wp})$. Then either $D_{\mathfrak{P}}$ is trivial and in this case Lemma follows from **(H.p1)**, or otherwise $D_{\mathfrak{P}}$ is a non-trivial p-group. Then,

$$\#H^0(M_{\mathfrak{P}}, X^*[\varpi]) = \#H^0\left(D_{\mathfrak{P}}, (H^0(M_{\mathfrak{P}}, X^*[\varpi])\right) \equiv \#H^0(F_{\wp}, X^*[\varpi]) \equiv 1 \mod p$$

where the last equality holds thanks to **(H.p1)** and local duality. This shows that $H^0(M_{\mathfrak{P}}, X^*) = 0$ and thus by local duality that $H^2(M_{\mathfrak{P}}, X) = 0$, as desired. The second assertion is proved in an identical manner. □

Definition 2.2. For $j = 0, 1, 2$ define

$$H^j_+(M_p, X) := \bigoplus_{i=1}^{s} \bigoplus_{\mathfrak{q}|\wp_i} H^j(M_{\mathfrak{q}}, X)$$

and

$$H^j_-(M_p, X) := \bigoplus_{i=1}^{s} \bigoplus_{\mathfrak{q}|\wp_i^c} H^j(M_{\mathfrak{q}}, X).$$

Proposition 2.3. *For M and X as above, the $\mathfrak{O}$-modules $H^1_+(M_p, X)$ and $H^1_-(M_p, X)$ are both free of rank $g \cdot d \cdot \delta_M$.*

Proof. Let $\mathfrak{P}$ be any prime of M above p. By [Nek06, Prop. 4.2.9], the cohomology $H^\bullet(M_{\mathfrak{P}}, X)$ is represented by a perfect complex of $\mathfrak{O}$-modules (i.e., projective (hence free) $\mathfrak{O}$-modules of finite type) concentrated in degrees $0, 1$ and 2. In particular, since we assume that $H^2(M_{\mathfrak{P}}, X) = 0$, then this complex may be taken in degrees 0 and 1. Similarly, the local cohomology $H^\bullet\left(M_{\mathfrak{P}}, \mathrm{Hom}_{\mathfrak{O}}(X, \mathfrak{O}(1))\right)$ is represented by a perfect complex of $\mathfrak{O}$-modules concentrated in degrees 0 amd 1. The two complexes $H^\bullet(M_{\mathfrak{P}}, X)$ and $H^\bullet\left(M_{\mathfrak{P}}, \mathrm{Hom}_{\mathfrak{O}}(X, \mathfrak{O}(1))\right)$ are related by the duality functor $\mathrm{RHom}_{\mathfrak{O}}(-, \mathfrak{O})[-2]$ (c.f., [Nek06, Prop. 5.2.4]). As a result, each of these two complexes is also represented by a perfect complex concentrated in degrees $2-1 = 1$ and $2 - 0 = 2$, hence by a single projective (hence free) $\mathfrak{O}$-module of finite type in degree 1. This shows that both $H^1_+(M_p, X)$ and $H^1_-(M_p, X)$ are free $\mathcal{O}$-modules of finite type.

Let M^{cyc}/M be the cyclotomic $\mathbb{Z}_p$-extension and let $\Lambda_M^{\mathrm{cyc}} = \mathfrak{O}[[\Gamma_M^{\mathrm{cyc}}]]$ be the completed group ring of its Galois group $\Gamma_M^{\mathrm{cyc}} = \mathrm{Gal}(M^{\mathrm{cyc}}/M)$. Let γ_M be a topological generator of Γ_M^{cyc}. Since the cohomological dimension of $G_{M_{\mathfrak{P}}}$ is 2 and since we assumed **(H.p1)**, it follows that

$$H^2(M_{\mathfrak{P}}, X \otimes \Lambda_M^{\mathrm{cyc}})/(\gamma_M - 1) \cong H^2(M_{\mathfrak{P}}, X) = 0,$$

and hence by Nakayama's lemma that

$$H^2(M_{\mathfrak{P}}, X \otimes \Lambda_M^{\mathrm{cyc}}) = 0. \tag{2.1}$$

Furthermore, as the ring Λ_M^{cyc} is Gorenstein, Nekovář's machinery applies as above *verbatim* for the Λ_M^{cyc}-module $X \otimes \Lambda_M^{\mathrm{cyc}}$ to conclude that the Λ_M^{cyc}-module $H^1(M_{\mathfrak{P}}, X \otimes \Lambda_M^{\mathrm{cyc}})$ is free of finite rank. Its rank equals $[M_{\mathfrak{P}} : \mathbb{Q}_p] \cdot d$ by [Büy09b, Theorem A.8(ii)]. The natural map

$$H^1(M_{\mathfrak{P}}, X \otimes \Lambda_M^{\mathrm{cyc}}) \longrightarrow H^1(M_{\mathfrak{P}}, X)$$

is surjective by (2.1). We conclude therefore that $H^1(M_{\mathfrak{P}}, X)$ is a free $\mathfrak{O}$-module of rank $[M_{\mathfrak{P}} : \mathbb{Q}_p] \cdot d$. This completes the proof. □

Lemma 2.4. *If* **(H.p1)** *holds true then the corestriction map*

$$\mathrm{cor} : H^1_{\pm}(M_p, X) \longrightarrow H^1_{\pm}(F_p, X)$$

is surjective.

Proof. Define $X_M = \mathrm{Ind}_{M/F} X$ and let $\mathfrak{A}_M$ be the augmentation ideal of the local ring $\mathfrak{O}[\Delta_M]$. By the semi-local Shapiro's Lemma [Rub00, §A.5], there is a canonical isomorphism

$$H^1_{\pm}(M_p, X) \cong H^1_{\pm}(F_p, X_M)$$

and the map

$$\mathrm{cor}_{M/F} : H^1_{\pm}(F_p, X_M) \cong H^1_{\pm}(M_p, X) \longrightarrow H^1_{\pm}(F_p, X)$$

is induced from the augmentation sequence

$$0 \longrightarrow \mathfrak{A}_M \cdot X_M \longrightarrow X_M \longrightarrow X \longrightarrow 0.$$

The cokernel of $\mathrm{cor}_{M/F}$ is therefore contained in $H^2_{\pm}(F_p, \mathfrak{A}_M \cdot X_M)$. To conclude the proof, it therefore suffices to check that $H^2(F_\wp, \mathfrak{A}_M \cdot X_M) = 0$ for every prime $\wp$ of F lying above p. This is equivalent by local duality to checking that $H^0(F_\wp, (\mathfrak{A}_M \cdot X_M)^*) = 0$. By the Claim on Page 1303 of [Büy10], it follows that $H^0(F_\wp, (\mathfrak{A}_M \cdot X_M)^*) \hookrightarrow H^0(F_\wp, X_M^*)$, hence the proof reduces to verify that $H^0(F_\wp, X_M^*) = 0$, which again by local duality is equivalent to the vanishing of $H^2(F_\wp, X_M) \cong H^2(M_{\mathfrak{P}}, X)$, and this is the conclusion of Lemma 2.1.

□

Proposition 2.5. *For M and X as above then the Λ_M-modules $H^1_{\pm}(M_p, X)$ are both free of rank $g \cdot d$.*

Proof. By Lemma 2.4, the map $H^1_{\pm}(M_p, X) \to H^1_{\pm}(F_p, X)$ (which may be thought of as reduction modulo the augmentation ideal $\mathfrak{A}_M \subset \mathfrak{O}[\Delta_M]$) is surjective. Nakayama's Lemma and Lemma 2.5 therefore imply that $H^1_{\pm}(M_p, X)$ is generated by (at most) $g \cdot d$ elements over the ring $\mathfrak{O}[\Delta_M]$. Let $\mathfrak{B} = \{x_1, x_2, \ldots, x_{g \cdot d}\}$ be any set of such generators. To prove (i), it suffices to check that the x_i's do not admit any non-trivial $\mathfrak{O}[\Delta_M]$-linear relation. Assume contrary, and suppose there is a non-trivial relation

$$\sum_{i=1}^{g \cdot d} \alpha_i x_i = 0, \quad \alpha_i \in \mathfrak{O}[\Delta_M]. \tag{2.2}$$

Write $S = \{\delta x_j : \delta \in \Delta_M, 1 \leq j \leq g \cdot d\}$, note that by our assumption on the set $\mathfrak{B}$, the set S generates $H^1_{\pm}(M_p, X)$ as an $\mathfrak{O}$-module, and $|S| = g \cdot d \cdot \delta_M = \mathrm{rank}_{\mathfrak{O}} H^1_{\pm}(M_p, X)$. Equation (2.2) may be rewritten as

$$\sum_{\delta, j} a_{\delta, j} \cdot \delta x_j = 0$$

with $a_{\delta,j} \in \mathfrak{O}$. Since we already know that $H^1(M_p, X)$ is $\mathfrak{O}$-torsion free, we may assume without loss of generality that $a_{\delta_0, j_0} \in \mathfrak{O}^\times$ for some δ_0, j_0. This in turn implies that

$$\delta_0 x_{j_0} \in \mathrm{span}_{\mathfrak{O}}(S - \{\delta_0 x_{j_0}\}),$$

hence $H^1(M_p, X)$ is generated by $S - \{\delta_0 x_{j_0}\}$. This, however, is a contradiction since we already know that the $\mathfrak{O}$-rank of $H^1(M_p, T)$ is $g \cdot d \cdot \delta_M = |S|$, hence it cannot be generated by $|S| - 1$ elements over $\mathfrak{O}$. The proof of the Proposition is now complete. □

Corollary 2.6.

(i) *The Λ-modules $H^1_{\pm}(F_p, X\otimes\Lambda)$ are both free of rank $g\cdot d$.*

(ii) *The $\Lambda_{\mathfrak{K}}$-modules $\varprojlim_{M\in\mathfrak{C}} H^1_{\pm}(M_p, X)$, where the inverse limits are with respect to corestriction maps, are both free of rank $g\cdot d$.*

Proof. Immediate after Proposition 2.5. □

When $\chi=\omega_\psi$, observe that $T=T_\chi\otimes\langle\psi\rangle^{-1}$. The hypothesis **(H.p1)** is verified for $X=T$ and $X=T_\chi$, since we assumed (1.2) and (1.3). In particular, the conclusions of Corollary 2.6 hold true both choices of G_F-representations.

Definition 2.7.

(1) Let $\mathfrak{L}$ be any free rank one $\Lambda_{\mathfrak{K}}$-direct summand of $\varprojlim_{M\in\mathfrak{C}} H^1_{+}(M_p, T_\chi)$.

(2) For $M=M_0\cdot F(\eta)\in\mathfrak{C}$ with $F\subset M_0\subset F_\infty$, let $\mathcal{L}_M\subset H^1_+(M_p,T_\chi)$ be the image of $\mathfrak{L}$ under the surjection
$$\varprojlim_N H^1(N_p,T_\chi)\to H^1(M_p,T_\chi).$$
We write $\mathcal{L}$ instead of $\mathcal{L}_F$.

(3) Let $\mathcal{L}_\infty$ be the image of $\mathfrak{L}$ under the surjection
$$\varprojlim_N H^1(N_p,T_\chi)\to H^1(F_p,T_\chi\otimes\Lambda).$$

Definition 2.8.

(1) The submodule
$$H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(F_p,T_\chi\otimes\Lambda)=H^1_-(F_p,T_\chi\otimes\Lambda)\oplus\mathcal{L}_\infty\subset H^1(F_p,T_\chi\otimes\Lambda)$$
is called the $\mathcal{L}_\infty$*-modified local condition* on $T_\chi\otimes\Lambda$.

(2) Similarly, the submodule
$$H^1_{\mathcal{F}_{\mathcal{L}}}(F_p,T_\chi)=H^1_-(F_p,T_\chi)\oplus\mathcal{L}\subset H^1(F_p,T_\chi)$$
is called the $\mathcal{L}$*-modified local condition* on T.

2.2. **Selmer structures.** The notation that we have set above is in effect.

We first recall Mazur and Rubin's definition of a *Selmer structure*, in particular the *canonical Selmer structure* on T_χ and $T_\chi\otimes\Lambda$ (or on their various twists).

Let R be a complete local noetherian $\mathfrak{O}$-algebra, and let X be a $R[[G_F]]$-module which is free of finite rank over R. In this paper, we will be interested in the case when $R=\Lambda$ or its certain quotients, and X is $T\otimes\Lambda$ or its relevant quotients by an ideal of Λ. (For example, taking the quotient by the augmentation ideal of Λ will give us $\mathfrak{O}$ and the representation T.)

Definition 2.9. A *Selmer structure* $\mathcal{F}$ on X is a collection of the following data:

- a finite set $\Sigma(\mathcal{F})$ of places of F, including all infinite places and primes above p, and all primes where X is ramified.
- for every $\lambda\in\Sigma(\mathcal{F})$ a local condition on X (which we view now as a $R[[\mathcal{D}_\lambda]]$-module), i.e., a choice of R-submodule
$$H^1_{\mathcal{F}}(F_\lambda,X)\subset H^1(F_\lambda,X).$$

If $\lambda \notin \Sigma(\mathcal{F})$ we will also write $H^1_{\mathcal{F}}(F_\lambda, X) = H^1_f(F_\lambda, X)$, where the module $H^1_f(F_\lambda, X)$ is the *finite* part of $H^1(F_\lambda, X)$, defined as in [MR04, Definition 1.1.6].

Definition 2.10. The *semi-local cohomology group* at a rational prime ℓ is defined by setting

$$H^i(F_\ell, X) := \bigoplus_{\lambda|\ell} H^i(F_\lambda, X).$$

Let λ be a prime of F. There is the perfect local Tate pairing

$$<\, ,\, >_\lambda : H^1(F_\lambda, X) \times H^1(F_\lambda, X^*) \longrightarrow H^2(F_\lambda, \mathfrak{F}/\mathfrak{O}(1)) \xrightarrow{\sim} \mathfrak{F}/\mathfrak{O},$$

where we recall that $X^* := \mathrm{Hom}(X, \boldsymbol{\mu}_{p^\infty})$ is the Cartier dual of X. For a Selmer structure $\mathcal{F}$ on X, define $H^1_{\mathcal{F}^*}(F_\lambda, X^*) := H^1_{\mathcal{F}}(F_\lambda, X)^\perp$ as the orthogonal complement of $H^1_{\mathcal{F}}(F_\lambda, X)$ with respect to the local Tate pairing. The Selmer structure $\mathcal{F}^*$ on X^* (with $\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}^*)$) defined in this way will be called the *dual Selmer structure*.

For examples of local conditions see [MR04, Definitions 1.1.6 and 3.2.1].

Definition 2.11. If $\mathcal{F}$ is a Selmer structure on X, we define the *Selmer module* $H^1_{\mathcal{F}}(F, X)$ as

$$H^1_{\mathcal{F}}(F, X) := \ker\left(H^1(\mathrm{Gal}(F_{\Sigma(\mathcal{F})}/F), X) \longrightarrow \bigoplus_{\lambda \in \Sigma(\mathcal{F})} H^1(F_\lambda, X)/H^1_{\mathcal{F}}(F_\lambda, X) \right),$$

where $F_{\Sigma(\mathcal{F})}$ is the maximal extension of F which is unramified outside $\Sigma(\mathcal{F})$. We also define the dual Selmer structure in a similar way; just replace X by X^* and $\mathcal{F}$ by $\mathcal{F}^*$ above.

Example 2.12. In this example we recall [MR04, Definitions 3.2.1 and 5.3.2] of which we make frequent use.

(i) Let $R = \mathfrak{O}$ and let X be a free R-module endowed with a continuous action of G_F, which is unramified outside a finite set of places of F. We define a Selmer structure $\mathcal{F}_{\mathrm{can}}$ on X by setting $\Sigma(\mathcal{F}_{\mathrm{can}}) = \{\lambda : X \text{ is ramified at } \lambda\} \cup \{\wp | p\} \cup \{v | \infty\}$, and
  - if $\lambda \in \Sigma(\mathcal{F}_{\mathrm{can}})$, $\lambda \nmid p\infty$, we define the local condition at λ to be
  $$H^1_{\mathcal{F}_{\mathrm{can}}}(F_\lambda, X) = \ker(H^1(F_\lambda, X) \longrightarrow H^1(F_\lambda^{\mathrm{unr}}, X \otimes \mathfrak{F})),$$
  where F_λ^{unr} is the maximal unramified extension of F_λ,
  - if $\wp | p$, we define the local condition at $\wp$ to be
  $$H^1_{\mathcal{F}_{\mathrm{can}}}(F_\wp, X) = H^1(F_\wp, X).$$

  The Selmer structure $\mathcal{F}_{\mathrm{can}}$ is called the *canonical Selmer structure* on X.

(ii) Let now R be the Iwasawa algebra Λ or its cyclotomic quotient Λ^{cyc}, and let $\mathbb{X}$ be a free R-module endowed with a continuous action of G_F, which is unramified outside a finite set of places of F. We define a Selmer structure $\mathcal{F}_R$ on $\mathbb{X}$ by setting
  $$\Sigma(\mathcal{F}_R) = \{\lambda : \mathbb{X} \text{ is ramified at } \lambda\} \cup \{\wp \subset F : \wp | p\} \cup \{v | \infty\},$$
  and $H^1_{\mathcal{F}_R}(F_\lambda, \mathbb{X}) = H^1(F_\lambda, \mathbb{X})$ for every $\lambda \in \Sigma(\mathcal{F}_R)$. The Selmer structure $\mathcal{F}_R$ is called the *canonical R-adic Selmer structure* on $\mathbb{X}$.

We still denote the induced Selmer structure on the quotients $\mathbb{X}/I\mathbb{X}$ by $\mathcal{F}_R$, which is obtained by *propagating* $\mathcal{F}_R$ on $\mathbb{X}$ (see [MR04, Example 1.1.2]). Note for $\lambda \in \Sigma(\mathcal{F}_R)$ that $H^1_{\mathcal{F}_R}(F_\lambda, \mathbb{X}/I\mathbb{X})$ will not always be the same as $H^1(F_\lambda, \mathbb{X}/I\mathbb{X})$. In particular, when I is the augmentation ideal of R, the Selmer structure $\mathcal{F}_R$ on $\mathbb{X}$ will not always propagate to $\mathcal{F}_{\mathrm{can}}$ on

$X := \mathbb{X} \otimes_R R/I$. However, when $X = T$ and $\mathbb{X} = T \otimes_{\mathfrak{O}} R$ as in §1, $\mathcal{F}_R$ on $\mathbb{X}$ *does* propagate to $\mathcal{F}_{\text{can}}$ on X, under the hypothesis $\mathbf{H.nE}$, as we shall check below.

Definition 2.13. A *Selmer triple* is a triple $(X, \mathcal{F}, \mathcal{P})$, where $\mathcal{F}$ is a Selmer structure on X and $\mathcal{P}$ is a set non-archimedean primes of F disjoint from $\Sigma(\mathcal{F})$.

Definition 2.14.

(a) Let $\mathcal{F}_-$ be the Selmer structure on $T_\chi \otimes \Lambda$ defined as follows:
  - $\Sigma(\mathcal{F}_-) = \Sigma(\mathcal{F}_\Lambda)$,
  - if $\lambda \nmid p$, define $H^1_{\mathcal{F}_-}(F_\lambda, T_\chi \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(F_\lambda, T_\chi \otimes \Lambda)$,
  - $H^1_{\mathcal{F}_-}(F_p, T_\chi \otimes \Lambda) := H^1_-(F_p, T_\chi \otimes \Lambda) \subset H^1(F_p, T_\chi \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(F_p, T_\chi \otimes \Lambda)$.

(b) Fix a Λ-rank one direct summand $\mathcal{L}_\infty \subset H^1_+(F_p, T_\chi \otimes \Lambda)$ as in Definition 2.7. Define the $\mathcal{L}_\infty$*-modified Selmer structure* $\mathcal{F}_{\mathcal{L}_\infty}$ on $T_\chi \otimes \Lambda$ as follows:
  - $\Sigma(\mathcal{F}_{\mathcal{L}_\infty}) = \Sigma(\mathcal{F}_\Lambda)$,
  - if $\lambda \nmid p$, define $H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(F_\lambda, T_\chi \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(F_\lambda, T_\chi \otimes \Lambda)$,
  - $H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(F_p, T_\chi \otimes \Lambda) := H^1_-(F_p, T_\chi \otimes \Lambda) \oplus \mathcal{L}_\infty \subset H^1_{\mathcal{F}_\Lambda}(F_p, T_\chi \otimes \Lambda)$.

2.3. **Comparing Selmer groups.** To ease notation, set $\mathbb{T} = T_\chi \otimes \Lambda$. Suppose in this section that $\chi = \omega_\psi$.

Lemma 2.15. *We have the following isomorphisms of* Λ*-modules:*

$$H^1(F, \mathbb{T}) \otimes \langle\psi\rangle^{-1} \cong H^1(F, T \otimes \Lambda), \tag{2.3}$$

$$H^1(F_\ell, \mathbb{T}) \otimes \langle\psi\rangle^{-1} \cong H^1(F_\ell, T \otimes \Lambda) \tag{2.4}$$

for every prime ℓ.

Proof. As we have remarked above, $T_\chi = T \otimes \langle\psi\rangle^{-1}$. As $\langle\psi\rangle^{-1}$ is a continuous character of Γ, the proof of the Lemma follows from [Rub00, Proposition 6.2.1].

□

Definition 2.16. Let $H^1_{\mathcal{F}_-}(F_p, T \otimes \Lambda) \subset H^1(F_p, T \otimes \Lambda)$ (resp., $\mathcal{L}^\psi_\infty$) be the isomorphic image of $H^1_{\mathcal{F}_-}(F_p, \mathbb{T})$ (resp., $\mathcal{L}_\infty$) under the isomorphism (2.4) above. For any subquotient X of $T \otimes \Lambda$, let $H^1_{\mathcal{F}_-}(F_\ell, X) \subset H^1(F_\ell, X)$ denote the propagated local condition on X, in the sense of [MR04, Example 1.1.2].

Lemma 2.17. *We have the following isomorphism of Selmer groups:*

$$H^1_{\mathcal{F}_-}(F, \mathbb{T}) \otimes \langle\psi\rangle^{-1} \cong H^1_{\mathcal{F}_-}(F, T \otimes \Lambda).$$

Proof. Immediate from Lemma 2.15 and the definition of the local condition $\mathcal{F}_-$ on $\mathbb{T}$ and on its twist $T \otimes \Lambda$. □

Lemma 2.18. *The* $\mathfrak{O}$*-module* $\mathcal{O}^{\times,\chi}_{L_\chi}$ *is free of rank* g.

Proof. This follows from [NSW08, §8.6.12], along with our assumption that χ is different from the Teichmüller character ω. □

Consider the following hypothesis:

$$H^1_{\mathcal{F}_-^*}(F, T_\chi^*) \text{ is finite.} \tag{2.5}$$

We will verify later (see Theorem 4.4 below) that (2.5) holds true, if we assume Leopoldt's conjecture.

Lemma 2.19. *Assuming* (2.5), $H^1_{\mathcal{F}_-}(F, T_\chi) = 0$.

Proof. Let $\mathcal{F}_{\text{can}}$ denote the canonical Selmer structure on T_χ, defined as in Example 2.12. As explained in [MR04, Lemma 6.1.2] and [Rub00, Proposition II.2.6], it follows from our assumption (1.3) that $H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi) \cong \mathcal{O}_{L_\chi}^{\times,\chi}$, and therefore $H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi)$ is an $\mathfrak{O}$-module of rank g by Lemma 2.18 under the running assumptions.

The Λ-module $H^1(F_p, T_\chi \otimes \Lambda)$ is free of rank $2g$ by Lemma 2.15 and Corollary 2.6, and $H^1_-(F_p, T_\chi \otimes \Lambda)$ is a free rank-g direct summand of $H^1(F_p, T_\chi \otimes \Lambda)$. Furthermore, the natural map

$$H^1(F_p, T_\chi \otimes \Lambda) \longrightarrow H^1(F_p, T_\chi)$$

is surjective thanks to the assumption (1.3). It thus follows that $H^1(F_p, T_\chi)$ is a free $\mathfrak{O}$-module of rank $2g$ and $H^1_{\mathcal{F}_-}(F_p, T_\chi)$ is a direct summand of this module of rank g. We conclude that

$$\text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_{\text{can}}}(F_p, T_\chi) - \text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_-}(F_p, T_\chi) = g.$$

Finally, it follows from [Wil95, Proposition 1.6] that

$$\begin{aligned}\Big(\text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi) - \text{corank}_{\mathfrak{O}}\, H^1_{\mathcal{F}^*_{\text{can}}}(F, T_\chi^*)\Big) - \Big(\text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_-}(F, T_\chi) - \text{corank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_-^*}(F, T_\chi^*)\Big) \\ = \text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_{\text{can}}}(F_p, T_\chi) - \text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_-}(F_p, T_\chi) \\ = g.\end{aligned}$$

Since $H^1_{\mathcal{F}^*_{\text{can}}}(F, T_\chi^*)$ is finite and $\text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi) = g$, we conclude that

$$\text{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_-}(F, T_\chi) = \text{corank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_-^*}(F, T_\chi^*).$$

Since we assumed (2.5), the proof follows. □

Proposition 2.20. *Assuming* (2.5), $H^1_{\mathcal{F}_-}(F, T \otimes \Lambda) = H^1_{\mathcal{F}_-}(F, \mathbb{T}) = 0$.

Proof. Let $\Gamma \cong \Gamma_1 \times \cdots \times \Gamma_{g+1-\delta}$, where $\Gamma_i \cong \mathbb{Z}_p$ and δ is Leopoldt's defect. Let γ_i be a topological generator of Γ_i. For each $1 \le j \le g+1-\delta$, set $\mathcal{A}_j = (\gamma_1 - 1, \cdots, \gamma_j - 1)$ and $\mathcal{A}_0 = 0$. There is an exact sequence

$$0 \longrightarrow T_\chi \otimes \Lambda/\mathcal{A}_{j-1} \xrightarrow{\gamma_j - 1} T_\chi \otimes \Lambda/\mathcal{A}_{j-1} \longrightarrow T_\chi \otimes \Lambda/\mathcal{A}_j \longrightarrow 0$$

that induces an injection

$$H^1_{\mathcal{F}_-}(F, T_\chi \otimes \Lambda/\mathcal{A}_{j-1}) \Big/ (\gamma_j - 1) \hookrightarrow H^1_{\mathcal{F}_-}(F, T_\chi \otimes \Lambda/\mathcal{A}_j).$$

Noting that

$$H^1_{\mathcal{F}_-}(F, T_\chi \otimes \Lambda/\mathcal{A}_{g+1-\delta}) = H^1_{\mathcal{F}_-}(F, T_\chi) = 0$$

and using Nakayama's Lemma at each step, it follows by induction that

$$H^1_{\mathcal{F}_-}(F, T_\chi \otimes \Lambda/\mathcal{A}_0) = H^1_{\mathcal{F}_-}(F, T_\chi \otimes \Lambda) = 0.$$

□

Proposition 2.21. *The following sequences of* Λ*-modules are exact:*

(i) $0 \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(k,\mathbb{T}) \stackrel{\mathrm{loc}_p^+}{\longrightarrow} \mathcal{L}_\infty \longrightarrow \Big(H^1_{\mathcal{F}_-^*}(k,\mathbb{T}^*)\Big)^\vee \longrightarrow \Big(H^1_{\mathcal{F}^*_{\mathcal{L}_\infty}}(k,\mathbb{T}^*)\Big)^\vee \longrightarrow 0.$

(ii) *For any class* $c \in H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(k,\mathbb{T})$,

$$0 \longrightarrow \frac{H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(k,\mathbb{T})}{\Lambda\cdot c} \stackrel{\mathrm{loc}_p^+}{\longrightarrow} \frac{\mathcal{L}_\infty}{\Lambda\cdot \mathrm{loc}_p^+(c)} \longrightarrow \Big(H^1_{\mathcal{F}_-^*}(k,\mathbb{T}^*)\Big)^\vee \longrightarrow \Big(H^1_{\mathcal{F}^*_{\mathcal{L}_\infty}}(k,\mathbb{T}^*)\Big)^\vee \longrightarrow 0.$$

Proof. The first follows exact sequence comes from Poitou-Tate global duality, used along with Proposition 2.20. The second is an immediate consequence of (i). □

3. RUBIN-STARK ELEMENTS AND AN EULER SYSTEM OF RANK r

In this section, we review Rubin's [Rub96] integral refinement of Stark's conjectures and construct Kolyvagin systems for the modified Selmer structure $\mathcal{F}_{\mathcal{L}_\infty}$ on $T_\chi\otimes\Lambda$, coming from the Rubin-Stark elements.

For the rest of this paper, we assume that the Rubin-Stark conjecture [Rub96, Conjecture B′] holds.

Let χ, f_χ and L be as above, and recall the definitions of the collections of extensions $\mathfrak{E}_0$ and $\mathfrak{E}$ from §1.1. Fix forever a finite set S of places of F that does *not* contain any prime above p, but contains the set of infinite places S_∞ and all primes $\lambda\nmid p$ at which χ is ramified. Assume that $|S|\geq r+1$. For each $\mathcal{K}\in\mathfrak{E}$, let

$$S_{\mathcal{K}} = \{\text{places of } \mathcal{K} \text{ that lie above the places in } S\}\cup\{\text{places of } \mathcal{K} \text{ at which } \mathcal{K}/F \text{ is ramified}\}$$

be a set of places of $\mathcal{K}$. Let $\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}$ denote the $S_{\mathcal{K}}$ units of $\mathcal{K}$, and $\Delta_{\mathcal{K}}$ (resp., $\delta_{\mathcal{K}}$) denote $\mathrm{Gal}(\mathcal{K}/F)$ (resp., $|\mathrm{Gal}(\mathcal{K}/F)|$). Rubin in [Rub96, Conjecture B′] predicts the existence of certain elements

$$\tilde{\varepsilon}_{\mathcal{K},S_{\mathcal{K}}} \in \Lambda_{\mathcal{K},S_{\mathcal{K}}} \subset \frac{1}{\delta_{\mathcal{K}}}\wedge^g\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}$$

where the module $\Lambda_{\mathcal{K},S_{\mathcal{K}}}$ is defined in [Rub96, §2.1] and has the property that for any homomorphism

$$\tilde{\psi}\in\mathrm{Hom}_{\mathbb{Q}_p[\Delta_{\mathcal{K}}]}(\wedge^g\mathcal{O}^{\times,\wedge}_{\mathcal{K},S_{\mathcal{K}}}\otimes\mathbb{Q}_p, \mathcal{O}^{\times,\wedge}_{\mathcal{K},S_{\mathcal{K}}}\otimes\mathbb{Q}_p)$$

which is induced from a homomorphism

$$\psi\in\mathrm{Hom}_{\mathbb{Z}_p[\Delta_{\mathcal{K}}]}(\wedge^g\mathcal{O}^{\times,\wedge}_{\mathcal{K},S_{\mathcal{K}}}, \mathcal{O}^{\times,\wedge}_{\mathcal{K},S_{\mathcal{K}}}),$$

one has $\tilde{\psi}(\Lambda_{K,S_K})\subset\mathcal{O}^{\times,\wedge}_{\mathcal{K},S_{\mathcal{K}}}$. We remark that the g-th exterior power $\wedge^g\mathcal{O}^{\times,\wedge}_{\mathcal{K},S_{\mathcal{K}}}$ (and all other exterior powers which appear below) is taken in the category of $\mathbb{Z}_p[\Delta_{\mathcal{K}}]$-modules.

Remark 3.1. Rubin's conjecture predicts that the elements $\tilde{\varepsilon}_{\mathcal{K},S_{\mathcal{K}}}$ should in fact lie inside the module $\frac{1}{\delta_{\mathcal{K}}}\wedge^g\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}},\mathcal{T}}$, where $\mathcal{T}$ is a finite set of primes disjoint from $S_{\mathcal{K}}$, chosen in a way that the group $\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}},\mathcal{T}}$ of $S_{\mathcal{K}}$-units which are congruent to 1 modulo all the primes in $\mathcal{T}$ is torsion-free. As explained in [Büy09b, Remark 3.1], one can safely ignore $\mathcal{T}$ as far as we are concerned in this paper.

Let F^{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of F and for $m\in\mathbb{Z}^+$, let F^{cyc}_m be the unique subextension of F of degree p^m.

Definition 3.2. For $\mathcal{K}=M\cdot L(\eta)\in\mathfrak{E}_0$ (or $\mathcal{K}=M\cdot F(\eta)\in\mathfrak{E}$), where $\eta\in\mathcal{N}(\mathcal{R})$ and $M\subset F_\infty$ a finite extension of F, choose $m\in\mathbb{Z}^+$ so that $M\not\subset F^{\mathrm{cyc}}_m$ and set $M_m=M\cdot F^{\mathrm{cyc}}_m$,

$\mathcal{K}_m = \mathcal{K} \cdot M_m$. Define

$$\varepsilon_{\mathcal{K},S_{\mathcal{K}}} = \mathbf{N}^r_{\mathcal{K}_m/\mathcal{K}} \left(\tilde{\varepsilon}_{\mathcal{K}_m, S_{\mathcal{K}_m}} \right)$$

where $\mathbf{N}^r_{\mathcal{K}_m/\mathcal{K}}$ denotes the norm map induced on the r-th exterior power. It follows from [Rub96, Proposition 6.1] that $\varepsilon_{\mathcal{K},S_{\mathcal{K}}}$ is well-defined.

As we have fixed S (therefore $S_{\mathcal{K}}$ as well), we will often drop S or $S_{\mathcal{K}}$ from the notation and denote $\varepsilon_{\mathcal{K},S_{\mathcal{K}}}$ by $\varepsilon_{\mathcal{K}}$; or sometimes use S instead of $S_{\mathcal{K}}$ and denote $\mathcal{O}_{\mathcal{K},S_{\mathcal{K}}}$ by $\mathcal{O}_{\mathcal{K},S}$.

For any number field $\mathcal{K}$, Kummer theory gives a canonical isomorphism

$$H^1(\mathcal{K}, \mathfrak{O}(1)) \cong \mathcal{K}^{\times,\wedge} \otimes_{\mathbb{Z}_p} \mathfrak{O} := \left(\varprojlim_n \mathcal{K}^\times / (\mathcal{K}^\times)^{p^n} \right) \otimes_{\mathbb{Z}_p} \mathfrak{O}.$$

Under this identification, we view each $\varepsilon_{\mathcal{K},S_{\mathcal{K}}}$ as an element of $\frac{1}{\delta_{\mathcal{K}}} \wedge^g H^1(\mathcal{K}, \mathfrak{O}(1))$. The distribution relation satisfied by the Rubin-Stark elements ([Rub96, Proposition 6.1]) shows that the collection $\{\varepsilon_{\mathcal{K},S_{\mathcal{K}}}\}_{K\in\mathcal{K}}$ is an Euler system of rank g in the sense of [PR98], as appropriately generalized by [Büy10] to allow denominators).

3.1. **Twisting by the character χ.** Following the formalism of [Rub00, §II.4], we may *twist* the Euler system $\{\varepsilon_{\mathcal{K},S_{\mathcal{K}}}\}_{\mathcal{K}\in\mathfrak{E}}$ of rank g for the representation $\mathfrak{O}(1)$, in order to obtain an Euler system for the representation $T_\chi = \mathfrak{O}(1) \otimes \chi^{-1}$.

For a finite sebextension M of F_∞/F and $\eta \in \mathcal{N}(\mathcal{R})$, let $\Gamma_M = \mathrm{Gal}(M/F)$ and define $\mathcal{K} = M \cdot F(\eta)$, $\mathcal{K}_0 = M \cdot L(\eta)$. Set $G^\eta := \mathrm{Gal}(F(\eta)/F)$, $\Delta^\eta := \mathrm{Gal}(L(\eta)/F) = G^\eta \times \Delta$, and finally $G^\eta_M := \mathrm{Gal}(\mathcal{K}/F) = G^\eta \times \Gamma_M$, which is the p-part of $\Delta^\eta_M := \mathrm{Gal}(\mathcal{K}_0/F) \cong G^\eta_M \times \Delta = G^\eta \times \Gamma_M \times \Delta$. (These canonical factorizations of the Galois groups follow easily from the fact that $|\Delta|$ is prime to p and from ramification considerations.) The array of fields and Galois groups below summarizes this paragraph:

$$\begin{tikzcd}
 & M\cdot L(\eta)=\mathcal{K}_0 \arrow[dl, dash, "\Delta"'] \arrow[dr, dash, "\Gamma_M"] \arrow[ddd, dash, "\Delta^\eta_M"] & \\
M\cdot F(\eta)=\mathcal{K} \arrow[d, dash, "\Gamma_M"'] \arrow[ddr, dash, "G^\eta_M"] & & L(\eta) \arrow[d, dash, "G^\eta"] \arrow[ddl, dash, "\Delta^\eta"'] \\
F(\eta) \arrow[dr, dash, "G^\eta"'] & & L \arrow[dl, dash, "\Delta"] \\
 & F &
\end{tikzcd}$$

Let χ be as above, and let ϵ_χ denote the idempotent $\frac{1}{|\Delta|}\sum_{\sigma\in\Delta} \chi(\sigma)\sigma^{-1}$, which regard as an element of the groups ring $\mathfrak{O}[\Delta^\tau_n]$ via the factorization above. For simplicity, we set $\delta = \delta_{\mathcal{K}}$ (note that δ also equals $\delta_{\mathcal{K}_0}$ up to multiplication by a p-adic unit) allowing ourselves to be somewhat sloppy, as the denominators will not be present when the Rubin-Stark elements are utilized for our main purposes.

For any integral ideal η which is prime to pf_χ, we define

$$\varepsilon_{\mathcal{K}_0}^{\chi} := \epsilon_\chi \varepsilon_{\mathcal{K}_0,S} \in \frac{1}{\delta}\epsilon_\chi \wedge^g H^1(\mathcal{K}_0, \mathfrak{O}(1)) \tag{3.1}$$

$$= \frac{1}{\delta} \wedge^g \epsilon_\chi H^1(\mathcal{K}_0, \mathfrak{O}(1)) \tag{3.2}$$

$$= \frac{1}{\delta} \wedge^g H^1(\mathcal{K}_0, \mathfrak{O}(1))^\chi. \tag{3.3}$$

Inflation-restriction yields

$$H^1(\mathcal{K}, \mathfrak{O}(1) \otimes \chi^{-1}) \longrightarrow H^1(\mathcal{K}_0, \mathfrak{O}(1) \otimes \chi^{-1})^\Delta.$$

On the other hand, since $G_{\mathcal{K}_0}$ is in the kernel of χ,

$$H^1(\mathcal{K}_0, \mathfrak{O} \otimes \chi^{-1}) \cong H^1(\mathcal{K}_0, \mathfrak{O}(1)) \otimes \chi^{-1},$$

hence

$$H^1(\mathcal{K}, T_\chi) \xrightarrow{\sim} H^1(\mathcal{K}_0, T_\chi)^\Delta \cong H^1(\mathcal{K}_0, \mathfrak{O}(1))^\chi.$$

This induces an isomorphism

$$\wedge^g H^1(\mathcal{K}, T) \xrightarrow{\sim} \wedge^g H^1(\mathcal{K}_0, \mathfrak{O}(1))^\chi. \tag{3.4}$$

The inverse image of the element $\varepsilon_{\mathcal{K}_0}^{\chi}$ (which was defined in (3.1)) under the isomorphism induced from (3.4) above will be denoted by $\varepsilon_{\mathcal{K}}^{\chi}$. The collection $\{\varepsilon_{\mathcal{K}}^{\chi}\}_{\mathcal{K}\in\mathfrak{E}}$ will be called the *Rubin-Stark element Euler system of rank r*.

Next, we construct an *Euler system of rank one* (i.e., an Euler system in the sense of [Rub00]) using ideas from [Rub96, §6] and [PR98, §1.2.3]. The main point is that, if one applied the arguments of [Rub96, PR98] directly, all one would get (after applying Kolyvagin's descent) would be a Λ-adic Kolyvagin system for the coarser Selmer structure $\mathcal{F}_\Lambda$ on $T_\chi \otimes \Lambda$. In Section §3.3, we overcome this difficulty and obtain a Λ-adic Kolyvagin system for the finer Selmer structure $\mathcal{F}_{\mathcal{L}_\infty}$ on $T_\chi \otimes \Lambda$.

3.2. **Choosing the homomorphisms.** For any field $\mathcal{K} \in \mathfrak{E}$, recall that $\Delta_\mathcal{K} := \mathrm{Gal}(\mathcal{K}/F)$ and write $\delta = |\Delta_\mathcal{K}|$. Using the elements of

$$\varprojlim_{\mathcal{K}\in\mathfrak{E}} \wedge^{r-1} \mathrm{Hom}_{\mathfrak{O}[\Delta_\mathcal{K}]}\left(H^1(\mathcal{K}, T_\chi), \mathfrak{O}[\Delta_K]\right) \tag{3.5}$$

(more precisely, using the elements those are in the image of the canonical map

$$\varprojlim_{\mathcal{K}\in\mathfrak{E}} \wedge^{r-1} \mathrm{Hom}_{\mathfrak{O}[\Delta_\mathcal{K}]}\left(H^1_+(\mathcal{K}_p, T_\chi), \mathfrak{O}[\Delta_\mathcal{K}]\right) \longrightarrow \varprojlim_{\mathcal{K}\in\mathfrak{E}} \wedge^{r-1} \mathrm{Hom}_{\mathfrak{O}[\Delta_\mathcal{K}]}\left(H^1(\mathcal{K}, T_\chi), \mathfrak{O}[\Delta_\mathcal{K}]\right) \tag{3.6}$$

which is induced from the localization followed by projection to $H^1_+(\mathcal{K}_p, T_\chi)$) and the Rubin-Stark elements above, we obtain an Euler system (in the sense of [Rub00]) for T_χ, following the arguments of [Büy10] (which are based on Rubin's ideas [Rub96, §6]; see also [PR98, §1.2.3]). We omit the details here and refer the reader to these articles.

Remark 3.3. For $\Psi = \{\psi_\mathcal{K}\} \in \varprojlim_{\mathcal{K}\in\mathfrak{E}} \wedge^{r-1} \mathrm{Hom}_{\mathfrak{O}[\Delta_\mathcal{K}]} (H^1(\mathcal{K}, T_\chi), \mathfrak{O}[\Delta_K])$ we have

$$\psi_\mathcal{K}(\varepsilon_{\mathcal{K}}^{\chi}) \in H^1(\mathcal{K}, T_\chi)$$

by the defining (integrality) property of the elements $\varepsilon_{\mathcal{K}}^{\chi} \in \frac{1}{\delta} \wedge^g H^1(\mathcal{K}, T_\chi)$, namely, the denominators δ will disappear once we apply the homomorphisms from (3.5) on the Rubin-Stark elements.

Let $\mathrm{ES}(T_\chi, \mathfrak{E}) = \mathrm{ES}(T_\chi)$ denote the collection of Euler systems for T_χ in the sense of [Rub00, §2] and [MR04, §3.2].

Definition 3.4. Let $\mathfrak{L} \subset \varprojlim_{M \in \mathfrak{E}} H^1_+(M_p, T_\chi)$ be a $\Lambda_{\mathfrak{K}}$-direct summand as in Definition 2.16. An Euler system $\mathbf{c} = \{c_{\mathcal{K}}\} \in \mathrm{ES}(T_\chi)$ is called an $\mathfrak{L}$-*restricted Euler system* if

$$\mathrm{loc}_p(c_{\mathcal{K}}) \in H^1_-(\mathcal{K}_p, T_\chi) \oplus \mathcal{L}_{\mathcal{K}}$$

for every $\mathcal{K} \in \mathfrak{E}$. The module of $\mathfrak{L}$-restricted Euler systems is denoted by $\mathrm{ES}_{\mathfrak{L}}(T_\chi)$.

The following Proposition is proved following the arguments of [Büy10, §3.3-3.4] *verbatim*. The key point is to make use of Corollary 2.6.

Proposition 3.5.

(i) *There exists* $\Psi = \{\psi_{\mathcal{K}}\} \in \varprojlim_{\mathcal{K} \in \mathfrak{E}} \wedge^{r-1} \mathrm{Hom}_{\mathfrak{O}[\Delta_{\mathcal{K}}]} \left(H^1_+(\mathcal{K}_p, T_\chi), \mathfrak{O}[\Delta_{\mathcal{K}}]\right)$ *such that* $\psi_{\mathcal{K}}$ *maps* $\wedge^r H^1_+(\mathcal{K}_p, T_\chi)$ *isomorphically onto* $\mathcal{L}_{\mathcal{K}}$.

(ii) *For* Ψ *as above, denote its image under* (3.6) *still by* Ψ. *Then*

$$\boldsymbol{c}^\chi_\Psi := \{\psi_{\mathcal{K}}(\varepsilon^\chi_{\mathcal{K}})\} \in \mathrm{ES}_{\mathfrak{L}}(T_\chi).$$

Let $c^\chi_{F,\Psi} := \psi_F(\varepsilon^\chi_F) \in H^1_{\mathcal{F}_{\mathcal{L}}}(F, T_\chi)$ be the initial term of the $\mathfrak{L}$-restricted Euler system $\mathbf{c}^\chi_\Psi$.

3.3. **From Euler systems of rank r to Kolyvagin systems for modified Selmer structures.** Let $\mathcal{P}_\chi$ be a fixed set of places of F that does not contain the archimedean places, primes at which T_χ is ramified and primes above p. Recall the definition of generalized module of Kolyvagin systems $\overline{\mathbf{KS}}(T_\chi \otimes \Lambda, \mathcal{F}_\Lambda, \mathcal{P}_\chi)$ from [MR04, Definition 3.1.6].

Theorem 3.6 (Mazur and Rubin)**.** *There is a canonical map*

$$\mathbf{ES}(T_\chi) \longrightarrow \overline{\mathbf{KS}}(T_\chi \otimes \Lambda, \mathcal{F}_\Lambda, \mathcal{P}_\chi),$$

with the property that if $\mathbf{c}$ *maps to* $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_\Lambda, \mathcal{P})$ *then*

$$\kappa_1 = \{c_M\} \in \varprojlim_M H^1(M, T_\chi) = H^1(F, T_\chi \otimes \Lambda),$$

where the inverse limit is over the finite sub-extensions M *of* F_∞/F.

Proof. Under the running hypotheses, this may be proved following the proof of Theorem 5.3.3 in [MR04] line by line. □

For Ψ and $\mathbf{c}^\chi_\Psi$ as in the statement of Proposition 3.5, we let $\boldsymbol{\kappa}^\chi \in \overline{\mathbf{KS}}(T_\chi \otimes \Lambda, \mathcal{F}_\Lambda, \mathcal{P}_\chi)$ be the image of $\mathbf{c}^\chi_\Psi$ under the Euler systems to Kolyvagin systems map of Theorem 3.6. The proof of [Büy10, Theorem 3.25] shows that:

Theorem 3.7. $\boldsymbol{\kappa}^\chi \in \overline{\mathbf{KS}}(T_\chi \otimes \Lambda, \mathcal{F}_{\mathcal{L}_\infty}, \mathcal{P}_\chi)$.

Remark 3.8. Although the author admits not to have checked details, he hopes that the arguments of [Büy09b, Theorem 2.19], [Büy11] and [Büy12a] would generalize to prove the existence of the Λ-adic Kolyvagin system $\boldsymbol{\kappa}^\chi$, which here we have constructed out of the conjectural Rubin-Stark elements.

4. Applications to the arithmetic of CM Abelian Varieties

Although our sights are set for applications of Theorem 3.7 on the study of CM abelian varieties, we first state the following two results, the latter of which may be thought of a generalization of Gras' conjecture. This result will later be used to convert all inequalities which are obtained using the Euler/Kolyvagin system machinery into equalities.

4.1. Ideal class groups of CM fields.

For any number field $\mathcal{K}$, let $A_{\mathcal{K}}$ denote the p-Sylow subgroup of the ideal class group of $A_{\mathcal{K}}$.

Theorem 4.1. $\#H^1_{\mathcal{F}^*_{\mathcal{L}}}(F,T^*_\chi) \leq [H^1_{\mathcal{F}_{\mathcal{L}}}(F,T_\chi) : \mathfrak{O}\cdot c^\chi_{F,\Psi}]$.

Proof. This follows from Theorem 3.7 and [MR04, Theorem 5.2.14]. □

We assume until the end the following is true (as well as the truth of Rubin-Stark conjectures):

H.LC Leopoldt's conjecture holds true for the field L_χ.

H.S The set S that appears in the definition of Rubin-Stark elements (see the start of §3) contains no archimedean places of F that splits in L_χ/F.

Theorem 4.2. (i) *The $\mathfrak{O}$-module $H^1_{\mathcal{F}_{\mathcal{L}}}(F,T_\chi)$ is free of rank one and $H^1_{\mathcal{F}^*_{\mathcal{L}}}(F,T^*_\chi)$ is finite.*

(ii) $\#A^\chi_{L_\chi} \leq [\wedge^g\,\mathcal{O}^{\times,\chi}_{L_\chi} : \mathfrak{O}\cdot\varepsilon^\chi_F]$.

(iii) *If the inequality in* (ii) *is sharp then so is the inequality in Theorem 4.1.*

Proof. (i) is proved using Theorem 4.1 and following the proof of [Büy09a, Corollary 3.6], and (ii) following the proof of [Büy09a, Theorem 3.10]. The key point in the proofs we refer to is that, the map ι from the group of units to semi-local units is injective, since Leopoldt's conjecture is assumed. To apply the arguments in loc.cit., one needs to replace the map ι by the map

$$\mathrm{loc}_p^+ : \mathcal{O}^{\times,\chi}_{L_\chi} \longrightarrow \left(\prod_{\mathfrak{P}\in\Sigma_p(L_\chi)} \mathcal{O}^\times_{L_\chi,\mathfrak{P}}\right)^\chi = H^1_+(F_p,T_\chi),$$

(where $\Sigma_p(L_\chi)$ is the collection of primes of L_χ that lie above the primes in Σ_p and the last equality follows from the proof of [Rub00, Proposition 2.6], since we assumed (1.3)) and check that loc_p^+ is injective under the running assumptions. This is what we verify now. Let Σ_{L_χ} denote the collection of infinite places of L_χ that lie above Σ and let $\Sigma'_{L_\chi}\subset\Sigma_{L_\chi}$ be any subset missing exactly one element $\sigma_0\in\Sigma_{L_\chi}$. Set $\mathfrak{r}=\#\Sigma'_{L_\chi}$ and let $\mathfrak{U}=\{\mathfrak{u}_1,\cdots,\mathfrak{u}_\mathfrak{r}\}$ be a set of fundamental system of units. The assumed hypothesis **H.LC** implies that the p-adic regulator

$$R_p(L_\chi) = \det\left(2\log_p(\iota_p\circ\sigma(\mathfrak{u}))\right)_{\mathfrak{u}\in\mathfrak{U};\,\sigma\in\Sigma'_{L_\chi}}$$

is non-zero. This in turn shows that the image of the map

$$\mathcal{O}^{\times,\wedge}_{L_\chi} \longrightarrow \prod_{\mathfrak{P}\in\Sigma_p(L_\chi)} \mathcal{O}^{\times,\wedge}_{L_\chi,\mathfrak{P}}$$

has rank $\mathfrak{r}$. Using the fact that the ring $\mathfrak{O}[\Delta]$ is semi-simple, it follows that the image of the map loc_p^+ is full rank. Since we assumed (1.4), we have $\chi\neq\omega$ and thus $\mathcal{O}^{\times,\chi}_{L_\chi}$ is torsion-free. It therefore follows that the map loc_p^+ is injective, as desired.

(iii) also follows from the proofs of Corollary 3.9 and Theorem 3.10 of loc.cit. □

Choosing the auxiliary set of primes $\mathcal{T}$ that appears in the definition of Rubin-Stark elements carefully (as in [Büy09a, §2.1], see also the discussion preceding Theorem 3.11 in loc.cit.), one may use the analytic class number formula for all the fields between L_χ and F to convert the inequality of Theorem 4.2(ii) (and therefore also in Theorem 4.1) into an equality. See [Rub92, §5] for further details.

Corollary 4.3. (i) $\#A_{L_\chi}^{\chi} = [\wedge^g \mathcal{O}_{L_\chi}^{\times,\chi} : \mathfrak{O}\cdot \varepsilon_F^{\chi}]$.
(ii) $\#H^1_{\mathcal{F}^*_{\mathcal{L}}}(F,T_\chi^*) = [H^1_{\mathcal{F}_{\mathcal{L}}}(F,T_\chi) : \mathfrak{O}\cdot c^{\chi}_{F,\Psi}]$

4.2. **Iwasawa theory.** Let $c^{\chi}_{F_\infty,\Psi} := \{c^{\chi}_{M,\Psi}\} \in \varprojlim_M H^1(M,T_\chi)$, where the inverse limit is over finite subextensions M of F_∞/F. Recall the Selmer structure $\mathcal{F}_-$ on $T_\chi\otimes\Lambda$, defined as in §2.3.

Let

$$\mathrm{loc}_p^{+,\otimes r} : \wedge^r H^1(F,X) \longrightarrow \wedge^r H^1_+(F_p,X)$$

(where $X = T_\chi$ or $T_\chi\otimes\Lambda$) be the map induced from loc_p^+. Define

$$\mathrm{loc}_p^{+,\otimes r}\left(\varepsilon^{\chi}_{F_\infty}\right) := \{\mathrm{loc}_p^{+,\otimes r}\left(\varepsilon^{\chi}_{M}\right)\}_M \in \varprojlim_M \wedge^r H^1_+(M,T_\chi) = \wedge^r H^1(F,T_\chi\otimes\Lambda),$$

where the inverse limit is with respect to finite subextensions M of F_∞/F, and the last equality holds true since the $\mathfrak{O}[\Gamma_M]$-module $H^1_+(M,T_\chi)$ is free of rank r, by Proposition 2.5.

Theorem 4.4. *Under the running assumptions,*

(i) *We have*

$$\#H^1_{\mathcal{F}^*_-}(F,T_\chi) = \#\mathcal{L}/\mathfrak{O}\cdot \mathrm{loc}_p^+\left(c^{\chi}_{F,\Psi}\right) = \#\wedge^r H^1_+(F_p,T_\chi)/\mathfrak{O}\cdot \mathrm{loc}_p^{+,\otimes r}\left(\varepsilon^{\chi}_F\right),$$

and all these quantities are finite.

(ii) $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathcal{L}_\infty}}(F,(T_\chi\otimes\Lambda)^*)^\vee\right) \mid \mathrm{char}\left(H^1_{\mathcal{F}_{\mathcal{L}_\infty}}(F,T_\chi\otimes\Lambda)/\Lambda\cdot c^{\chi}_{F_\infty,\Psi}\right)$.

(iii) *The module $H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)$ is Λ-cotorsion and*

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)^\vee\right) = \mathrm{char}\left(\wedge^r H^1_+(F_p,T_\chi\otimes\Lambda)/\Lambda\cdot \mathrm{loc}_p^{+,\otimes r}\left(\varepsilon^{\chi}_{F_\infty}\right)\right)$$

Proof. The second equality in (i) is deduced using the defining property of Ψ, see Proposition 3.5(i).

As in Proposition 2.21[1], the Poitou-Tate global duality and Lemma 2.19 yields an exact sequence

$$0\to H^1_{\mathcal{F}_{\mathcal{L}}}(F,T_\chi)/\mathfrak{O}\cdot c^{\chi}_{F,\Psi} \longrightarrow \mathcal{L}/\mathfrak{O}\cdot \mathrm{loc}_p^+\left(c^{\chi}_{F,\Psi}\right) \longrightarrow H^1_{\mathcal{F}^*_-}(F,T_\chi^*) \longrightarrow H^1_{\mathcal{F}^*_{\mathcal{L}}}(F,T_\chi^*) \to 0$$

The first equality in (i) now follows from Corollary 4.3(ii).

When the Iwasawa algebra Λ has Krull dimension 2, (ii) follows from [MR04, Theorem 5.3.1]. The general case is reduced to the case of dimension 2 applying the results of Ochiai in [Och05, §3].

Let $\mathcal{A} = \ker\{\Lambda \to \mathfrak{O}\}$ be the augmentation ideal. By [MR04, Lemma 3.5.3], we may identify $H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)[\mathcal{A}]$ with $H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*[\mathcal{A}])$, and

$$\begin{aligned} H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*[\mathcal{A}]) &= H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda/\mathcal{A})^*) \\ &= H^1_{\mathcal{F}^*_-}(F,T_\chi^*). \end{aligned} \tag{4.1}$$

[1]See also [Rub00] Theorem I.7.3, proof of Theorem III.2.10 and [dS87, §III.1.7].

Since $H^1_{\mathcal{F}^*_-}(F, T^*_\chi)$ is finite by (i), the first statement of (iii) follows. It also follows from (ii) and Proposition 2.21(ii) (along with the choice of Ψ as in Proposition 3.5) that

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)^\vee\right) \mid \mathrm{char}\left(\wedge^r H^1_+(F_p,T_\chi\otimes\Lambda)/\Lambda\cdot\mathrm{loc}_p^{+,\otimes r}\left(\varepsilon^\chi_{F_\infty}\right)\right).$$

It remains to prove that this divisibility may in fact be turned into an equality and this is what we carry out in what follows.

We will first check the equality modulo the augmentation ideal $\mathcal{A}$ (namely, the statements (4.2) and (4.3)) , which using Lemma 4.6, Lemma 4.7 below and the divisibility obtained above will conclude the proof.

Observe that the Λ-module $\mathcal{L}_\infty/\Lambda\cdot\mathrm{loc}_p^+(c^\chi_{F_\infty,\Psi})$ is cyclic. We see therefore that

$$\begin{aligned}\mathrm{char}\left(\wedge^r H^1_+(F_p,T_\chi\otimes\Lambda)/\Lambda\cdot\mathrm{loc}_p^{+,\otimes r}\left(\varepsilon^\chi_{F_\infty}\right)\right) &= \mathrm{char}\left(\mathcal{L}_\infty/\Lambda\cdot\mathrm{loc}_p^+(c^\chi_{F_\infty,\Psi})\right)\\ &= \mathrm{Fitt}_\Lambda\left(\mathcal{L}_\infty/\Lambda\cdot\mathrm{loc}_p^+(c^\chi_{F_\infty,\Psi})\right),\end{aligned}$$

where the first equality is obtained thanks to the choice of Ψ and $\mathrm{Fitt}_\Lambda(M)$ denotes the initial Fitting ideal of a Λ-module M. Thus,

$$\begin{aligned}[\mathfrak{O}:\mathrm{char}(\mathcal{L}_\infty/\Lambda\cdot c^\chi_{F_\infty,\Psi})\otimes_\Lambda\Lambda/\mathcal{A}] &= [\mathfrak{O}:\mathrm{Fitt}_{\mathfrak{O}}\left((\mathcal{L}_\infty/\Lambda\cdot c^\chi_{F_\infty,\Psi})\otimes_\Lambda\Lambda/\mathcal{A}\right)]\\ &= [\mathfrak{O}:\mathrm{Fitt}_{\mathfrak{O}}(\mathcal{L}/\mathfrak{O}\cdot c^\chi_{F,\Psi})]\\ &= \#H^1_{\mathcal{F}^*_-}(F,T_\chi).\end{aligned}\tag{4.2}$$

We next check that

$$\begin{aligned}[\mathfrak{O}:\mathrm{char}\left(H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)^\vee\right)\otimes_\Lambda\Lambda/\mathcal{A}] &= \#H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)^\vee\otimes\Lambda/\mathcal{A}\\ &= \#H^1_{\mathcal{F}^*_-}(F,T_\chi).\end{aligned}\tag{4.3}$$

where the second equality holds thanks to (4.1). In order to achieve this, we appeal to Nekovář's theory of Selmer complexes and the descent formalism built in his theory.

The Selmer complex $\widetilde{R\Gamma}_{f,\mathrm{Iw}}(F_\infty/F,T_\chi)$ (resp., the dual complex $\widetilde{R\Gamma}_f(F_{\Sigma(\mathcal{F}_{\mathrm{can}})}/F_\infty,T^*_\chi)$, in the sense of [Nek06, Proposition 9.7.2]) related to the Selmer group $H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)$ is given by the Greenberg local conditions determined by

$$U_v^+(T_\chi)=\begin{cases}0, & \text{if } v\in\Sigma_p,\\ T_\chi, & \text{if } v\in\Sigma_p^c.\end{cases}$$

(resp.,

$$U_v^+(T^*_\chi)=\begin{cases}T^*_\chi, & \text{if } v\in\Sigma_p,\\ 0, & \text{if } v\in\Sigma_p^c.\end{cases}$$

for the dual complex). Since we assume (1.3), [Nek06, Lemma 9.6.3] (and [Nek06, Proposition 8.8.6] to pass to limit) shows that

$$\tilde{H}^1_f(F_{\Sigma(\mathcal{F}_{\mathrm{can}})}/F_\infty,T^*_\chi)\xrightarrow{\sim}H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*),$$

and [Nek06, 8.9.6.2] that

$$\tilde{H}^2_{f,\mathrm{Iw}}(F_\infty/F,T_\chi)\cong\tilde{H}^1_f(F_{\Sigma(\mathcal{F}_{\mathrm{can}})}/F_\infty,T^*_\chi)^\vee.$$

Furthermore, we proved above that the Λ-module

$$\tilde{H}^1_f(F_{\Sigma(\mathcal{F}_{\mathrm{can}})}/F_\infty,T^*_\chi)\cong H^1_{\mathcal{F}^*_-}(F,(T_\chi\otimes\Lambda)^*)$$

is cotorsion. Set $A_\chi = T_\chi \otimes \mathfrak{F}/\mathfrak{O}$ and fix a prime $\wp$ of F and a prime $\tilde{\wp}$ of L_χ above $\wp$. Let $\Delta_\wp$ be the decomposition group of $\tilde{\wp}$ in Δ. Since we assumed (1.3) and (1.4), we observe that (see [Nek06, §9.5])

$$\tilde{H}^0_f(F, A_\chi) = \boldsymbol{\mu}_{p^\infty}(L_\chi)^\chi = 0 \tag{4.4}$$

$$\tilde{H}^3_f(F, T_\chi) = (\mathfrak{O}[\Delta/\Delta_\wp])^\chi = 0 \tag{4.5}$$

hence by duality that

$$\tilde{H}^0_f(F, X) = 0$$

for $X = A_\chi, T^*_\chi$. Thus the proof of [Nek06, Proposition 9.7.7] shows that the complex $\widetilde{R\Gamma}_{f,\mathrm{Iw}}(F_\infty/F, T_\chi)$ may be represented by a complex

$$\mathrm{Cone}\left(M \xrightarrow{u} M\right)[-2]$$

where M is a free Λ-module of finite type and u is injective. This, together with Nekovář's control theorem [Nek06, Proposition 8.10.1]

$$\widetilde{R\Gamma}_{f,\mathrm{Iw}}(F_\infty/F, T_\chi) \overset{\mathbf{L}}{\otimes}_\Lambda \mathfrak{O} \xrightarrow{\sim} \widetilde{R\Gamma}_f(F, T_\chi)$$

concludes the proof of (4.3), hence also the proof of the theorem. □

Remark 4.5. One may also observe directly by Proposition 2.20 that

$$\tilde{H}^1_{f,\mathrm{Iw}}(F_\infty/F, T_\chi) \cong H^1_{\mathcal{F}_-}(F, T_\chi \otimes \Lambda) = 0$$

and thus the Selmer complex $\widetilde{R\Gamma}_{f,\mathrm{Iw}}(F_\infty/F, T_\chi)$ has no cohomology in degree 1. Also, the vanishing (4.5) implies by Nakayama's Lemma (using [Nek06, 8.10.3.3]) that the Selmer complex has no cohomology in degree 3 either.

Lemma 4.6. char $\left(H^1_{\mathcal{F}^*_-}(F, (T_\chi \otimes \Lambda)^*)^\vee\right) \not\subset \mathcal{A}$.

Proof. This follows from (4.3) and the finiteness of $H^1_{\mathcal{F}^*_-}(F, T^*_\chi)$. □

Lemma 4.7. *Suppose $f, g \in \Lambda$ are such that $f \mid g$, $f - g \in \mathcal{A}$ and $f \notin \mathcal{A}$. Then $f/g \in \Lambda^\times$.*

Proof. Write $g = f \cdot h$ with $h \in \Lambda$, so that $f - g = f(1-h) \in \mathcal{A}$. Since $f \notin \mathcal{A}$, it follows that $1 - h \in \mathcal{A} \subset \mathfrak{m}_\Lambda$, where $\mathfrak{m}_\Lambda$ is the maximal ideal. Hence h is indeed a unit. □

Remark 4.8. Mimicking the proof of [Rub00, Proposition 3.2.6] (with the aid of the assumption (1.3)), we see that

$$H^1_+(F_p, T_\chi \otimes \Lambda) = \varprojlim_M \bigoplus_{\wp \in \Sigma_p} \mathcal{O}^{\times,\chi}_{ML_\chi,\wp}$$

where $\mathcal{O}_{ML_\chi,\wp} = \mathcal{O}_{ML_\chi} \otimes \mathcal{O}_{F_\wp}$ and the inverse limit is over finite subextensions of F_∞/F. Similarly (using again the arguments of [Rub00, §1.6]), the Λ-module $H^1_{\mathcal{F}^*_-}(F, (T_\chi \otimes \Lambda)^*)$ may be identified by $\mathrm{Gal}(M_\infty/L_\infty)^\chi$, where $L_\infty = L_\chi F_\infty$ and M_∞ is the maximal abelian extension of L_∞ unramified outside Σ_p. Theorem 4.4(iii) may therefore be regarded as a natural generalization of [Rub91, Theorem 4.1].

4.3. **Katz's p-adic L-function.** Attached to the p-ordinary CM-type Σ and the character χ, Katz [Kat78] and Hida-Tilouine [HT93, Theorem II] has constructed a p-adic L-function $\mathcal{L}_\chi^\Sigma \in \Lambda_{\mathcal{W}} := \mathcal{W}_{\mathfrak{O}}[[\Gamma]]$, where $\mathcal{W}_{\mathfrak{O}}$ is the compositum of $\mathfrak{O}$ and $\mathcal{W}$ (and $\mathcal{W}$ is the p-adic completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$), that p-adically interpolates the algebraic parts (in the sense of [Shi75]) of the critical Hecke L-values for χ twisted by the characters of Γ. In the spirit of the main result of [Yag82], we propose the following conjecture under our running assumptions:

Conjecture 3. char $\left(\wedge^g H^1_+(F_p, T_\chi \otimes \Lambda_{\mathcal{W}})/\Lambda_{\mathcal{W}} \cdot \mathrm{loc}_p^{+,\otimes g}\left(\varepsilon^\chi_{F_\infty}\right)\right) = \left(\mathcal{L}_\chi^\Sigma\right)$.

In view of Theorem 4.4(iii), this conjecture is equivalent to the following statement:

Theorem 4.9. *Conjecture 1 holds true if and only if the Katz p-adic L-function $\mathcal{L}_\chi^\Sigma$ generates* char $\left(H^1_{\mathcal{F}_-^*}(F, (T_\chi \otimes \Lambda_{\mathcal{W}})^*)^\vee\right)$.

The latter statement is known as the $(g+1)$-variable[2] main conjecture, see [HT94, Page 90]. Building on the works of Hida-Tilouine [HT94], Hida [Hid06, Hid09], Mainardi [Mai08], Hsieh has proved the CM main conjecture in [Hsi12] under the following hypotheses:

(**H.1**) $p > 5$ is prime to the minus part of the class number of F, to the order of χ and is unramified in $K/\mathbb{Q}$.

(**H.2**) χ is unramified in Σ_p^c and $\chi\omega^{-a}$ is unramified at Σ_p for some integer $a \not\equiv 2 \mod p-1$.

(**H.3**) χ is *anticyclotomic* in the sense that $\chi(c\delta c^{-1}) = \chi(\delta)^{-1}$ for $\delta \in \Delta$ and $c \in G_K$ that induces the generator of Gal(F/K).

(**H.4**) $\chi(\wp) \neq 1$ for any $\wp \in \Sigma_p$. (Compare to (1.3))

(**H.5**) The restriction of χ to $G_{F(\sqrt{p^*})}$ (where $p^* = (-1)^{\frac{p-1}{2}}p$) is non-trivial.

The g-variable *anticyclotomic* main conjecture (and therefore the g-variable version of Conjecture 1 above) may be verified under much less restrictive hypothesis, namely assuming only **H.3-5**. See [Hid09, Corollary 2].

4.4. **Hecke characters attached to CM abelian varieties.** We start this subsection with an overview of well-known facts about CM abelian varieties that we need below, which are originally due to Serre-Tate and Shimura. Let $A_{/K}$ be an abelian variety which has CM by F. We assume that End$_F(A) = \mathcal{O}_F$; however, the arguments in this section will carry out to the more general case when the index of the order End$_F(A)$ inside the maximal order $\mathcal{O}_F$ is assumed to be prime to p. Assume also that the field F contains no nontrivial p-th root of unity.

Let $T_p(A) = \varprojlim A[p^n]$ be the p-adic Tate-module of A. It is a free $\mathbb{Z}_p$-module of rank $2g$ on which G_F acts continuously. As explained in the Remark on page 502 of [ST68], $T_p(A)$ is free of rank one over $\mathcal{O}_F \otimes \mathbb{Z}_p = \prod_\wp \mathcal{O}_\wp$, where the product is over the primes of F that lie above p. This yields a decomposition $T_p(A) = \bigoplus_\wp T_\wp(A)$, where each $T_\wp(A) = \varprojlim A[\wp^n]$ is a free $\mathcal{O}_\wp$-module of rank one. The G_F-action on $T_p(A)$ gives rise to a character

$$\psi_\wp : G_F \longrightarrow \mathcal{O}_\wp^\times.$$

By [Rib76, §2], $\psi_\wp$ is surjective for p large enough; we fix until the end a prime p satisfying this condition. We thence obtain a decomposition

$$T_p(A) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p = \bigoplus_{\wp | p} \bigoplus_{\sigma : F_\wp \hookrightarrow \overline{\mathbb{Q}}_p} V_\wp^\sigma$$

[2]As we assumed **(H.LC)**, the Iwasawa algebra $\Lambda = \mathfrak{O}[[\Gamma]]$ is isomorphic to a power series ring in $g+1$ variables.

where $V_\wp^\sigma$ is the one-dimensional $\overline{\mathbb{Q}}_p$-vector space on which G_F acts via the character $\psi_\wp^\sigma$, which is the compositum

$$G_F \xrightarrow{\psi_\wp} \mathcal{O}_\wp^\times \xrightarrow{\sigma} \overline{\mathbb{Q}}_p$$

Fix an embedding $j_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $j_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ extending ι_p. Let $\mathfrak{J} = \Sigma \cup \Sigma^c$ be the set of all embeddings of F into $\overline{\mathbb{Q}}$. Attached to A, there is a character

$$\boldsymbol{\psi} : \mathbb{A}_F/F^\times \longrightarrow F^\times,$$

which induces the Grössencharacters

$$\psi_\tau = j_\infty \circ \tau \circ \boldsymbol{\psi} : \mathbb{A}_F/F^\times \longrightarrow \mathbb{C}^\times$$

and its p-adic avatars

$$\psi_\tau^{(p)} = j_p \circ \tau \circ \boldsymbol{\psi} : \mathbb{A}_F/F^\times \longrightarrow \mathbb{C}_p^\times.$$

Theory of complex multiplication identifies the two sets $\{\mathrm{rec} \circ \psi_\tau^{(p)}\}_{\tau\in\mathfrak{J}}$ and $\{\psi_\wp^\sigma\}_{\wp,\sigma}$ of p-adic Hecke characters, where $\mathrm{rec} : \mathbb{A}_F/F^\times \to G_F$ is the reciprocity map. The Hasse-Weil L-function $L(A/F, s)$ of A then factors into a product of Hecke L-series

$$L(A/F, s) = \prod_{\tau\in\mathfrak{J}} L(\psi_\tau, s).$$

We assume henceforth that A is principally polarized. Fix $\varepsilon \in \Sigma$ and identify F by F^ε. This choice in turn fixes a prime $\wp \in \Sigma_p$ and $\sigma : F_\wp \hookrightarrow \overline{\mathbb{Q}}_p$ in a way that $\mathrm{rec} \circ \psi_\varepsilon^{(p)} = \psi_\wp^\sigma$. Set $\mathfrak{O} := \sigma(\mathcal{O}_{F_\wp})$ and define

$$\psi := \psi_\wp^\sigma : G_F \twoheadrightarrow \mathfrak{O}^\times;$$

this is the Hecke character for which we apply the results from §4.3. Note in particular that we have $T^* \cong A[\varpi^\infty]$. For $\wp$ as above, we assume the following non-anomaly condition on A:

(4.6) $$A(F_v)[\varpi] = 0 \text{ for every prime } v \text{ of } F \text{ above } p.$$

For the character ψ, the condition (1.2) holds trivially true. The hypothesis (1.3) holds true for $\chi = \omega_\psi$ since we assumed (4.6). Since we have assumed that F contains no p-th roots of unity, it follows that the Teichmüller character ω is totally ramified at all primes above p; and as ω_ψ is ramified at only $\wp$, the condition (1.4) for $\chi = \omega_\psi$ is also satisfied.

Let $g_\chi \in \mathrm{char}\left(H^1_{\mathcal{F}^*_-}(F, \mathbb{T}^*)^\vee\right)$ be any generator. Assuming Conjecture 3, we have $g_\chi = u \cdot \mathcal{L}_\chi^\Sigma$ for a unit $u \in \Lambda_{\mathcal{W}}^\times$.

Theorem 4.10. *Assume the truth of Conjecture 3 and the hypotheses of Theorem 4.4.*

(i) *For* $\alpha_\varpi = \mathrm{ord}_\varpi(\psi(g_\chi))$, *we have* $|H^1_{\mathcal{F}^*_-}(F, T^*)| = p^{\alpha_\wp}$.

(ii) *Suppose* $L(\psi_\varepsilon, 0) \neq 0$. *Then* $A(F)$ *is finite and* $Ш_{A/F}[\varpi^\infty]$ *is finite.*

Proof. Let $A_\psi = \ker\left\{\Lambda \xrightarrow{\gamma\mapsto\psi(\gamma)} \mathfrak{O}\right\}$. A variant of the proof of the identity (4.3) shows that

$$[\mathfrak{O} : \mathrm{char}\left(H^1_{\mathcal{F}^*_-}(F, (T_\chi \otimes \Lambda)^*)^\vee\right) \otimes_\Lambda \Lambda/\mathcal{A}_\psi] = |H^1_{\mathcal{F}^*_-}(F, T^*)|$$

and that

$$\begin{aligned}[\mathfrak{O} : \mathrm{char}\left(H^1_{\mathcal{F}^*_-}(F, (T_\chi \otimes \Lambda)^*)^\vee\right) \otimes_\Lambda \Lambda/\mathcal{A}_\psi] &= [\mathfrak{O} : \psi\left(\mathrm{char}(H^1_{\mathcal{F}^*_-}(F, (T_\chi \otimes \Lambda)^*)^\vee)\right)] \\ &= [\mathfrak{O} : \psi(g_\chi)]\end{aligned}$$

This proves (i). Let $\mathcal{F}_{\mathrm{BK}}$ (resp., $\mathcal{F}^*_{\mathrm{BK}}$) denote the Bloch-Kato Selmer structure on T (resp., on T^*), defined as in [BK90] and [Büy10, §2.3.1]. We observe that,

- for any finite subextention $\mathcal{M}$ of F_∞ and a prime $\mathfrak{p}$ of $\mathcal{M}$ above p, we have
$$H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathcal{M}_\mathfrak{p}, T_\chi) = \mathcal{U}^\chi_{L_\chi M} = H^1(\mathcal{M}_\mathfrak{p}, T_\chi),$$
where $\mathcal{U}^\chi_{L_\chi M}$ are the χ-part of the semi-local units of $L_\chi M$ at $\mathfrak{p}$ of M above p, and where the last equality holds thanks to the assumption (4.6);

- for a place $\lambda \nmid p$ of F,
$$\begin{aligned} H^1_{\mathcal{F}_{\mathrm{BK}}}(F_\lambda, T) &\supset H^1_{\mathrm{ur}}(F_\lambda, T) \\ &\supset \mathrm{im}\left(H^1(F_\lambda, T\otimes\Lambda^{\mathrm{cyc}}) \to H^1(F_\lambda, T)\right) \\ &\supset \mathrm{im}\left(H^1(F_\lambda, T\otimes\Lambda) \to H^1(F_\lambda, T)\right) \\ &=: H^1_{\mathcal{F}_-}(F_\lambda, T), \end{aligned}$$

where the first inclusion follows from [Rub00, Lemma I.3.5(ii)] and second from [Rub00, Corollary B.3.4]. We thus conclude on the dual side that $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(F, T^*) \subset H^1_{\mathcal{F}^*_-}(F, T^*)$, hence (ii) follows from (i) using the exact sequence
$$0 \longrightarrow A(F)\otimes \mathfrak{F}/\mathfrak{O} \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{BK}}}(F, T^*) \longrightarrow Ш_{A/F}[\varpi^\infty] \longrightarrow 0,$$
along with the interpolation property of $\mathcal{L}^\Sigma_\chi$; see [HT93, Theorem II]. $\square$

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KÂZIM BÜYÜKBODUK
KOÇ UNIVERSITY, MATHEMATICS
RUMELI FENERI YOLU, 34450 SARIYER
ISTANBUL, TURKEY