On the anticyclotomic Iwasawa theory of CM forms at supersingular primes

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ABSTRACT. In this paper, we study the anticyclotomic Iwasawa theory of a CM form f of even weight $w \geq 2$ at a supersingular prime, generalizing the results in weight 2, due to Agboola and Howard. In due course, we are naturally lead to a conjecture on universal norms that generalizes a theorem of Perrin-Riou and Berger and another that generalizes a conjecture of Rubin (which seems ultimately linked to the local divisibility of Heegner points). Assuming the truth of these conjectures, we establish a formula for the variation of the sizes of the Selmer groups attached to the *central critical twist* of f as one climbs up the anticyclotomic tower. We also prove a statement which may be regarded as a form of the anticyclotomic main conjecture (without p-adic L-functions) for the central critical twist of f.

CONTENTS

1. INTRODUCTION

The goal of this article is to study the anticyclotomic Iwasawa theory of an elliptic newform f of even weight $w \geq 2$ which has CM by an imaginary quadratic field K, at a supersingular prime p and thereby extend the results of Agboola and Howard [AH05] where the authors have studied the case $w = 2$. The results of this paper also should be considered as the first steps

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towards an anticyclotomic main conjecture for CM forms in the supersingular setting, where the analogous results when p is an ordinary prime for f have been obtained by Arnold [Arn07].

There are two main ingredients that go into the proof of the results presented in this article:

(1) The construction and analysis of $\pm$-Selmer groups, in the spirit of Kobayashi [Kob03].
(2) The proof that a *central critical twist* of the elliptic unit Euler system along the anticyclotomic Iwasawa tower is non-vanishing; and this Euler system controls the $\pm$-Selmer groups constructed in step (1).

Although our approach relies (generally speaking) on the arguments of [AH05], one runs into serious difficulty if (s)he attempts to carry out the steps (1) and (2) above when $w \neq 2$. Let us explain the difficulties which one encounters and sketch how we attempt to resolve them in this paper.

First of all, when the weight is 2, Agboola and Howard have the formal group of the associated elliptic curve at their disposal to analyze the local cohomology groups. Their approach does not apply in the general setting, for which reason one is lead to appeal to p-adic Hodge Theory to carry out the step (1) above; see §2 below. The desired properties for the $\pm$-subgroups of the local cohomology groups all then follow from suitable generalizations (which unfortunately we are unable to prove) of the following fundamental results: The first (Conjecture 2.5) is a statement about the universal norms along the anticyclotomic tower and an extension of a conjecture of Nekovář (on the universal norms along the cyclotomic tower) which was proved by Perrin-Riou [PR00] and Berger [Ber05]; also proved by Rubin [Rub85] along the anticyclotomic tower when the weight is 2. The second (Conjecture 2.13) is a generalization of [Rub87, Conj. 2.2] and Rubin proved his conjecture when the weight is 2 under some assumptions on the prime p, by relating it to the local divisibility of Heegner points. It would be interesting to know if Nekovář's [Nek92] Heegner cycles on Kuga-Sato varieties could play a role along these line to prove Conjecture 2.13 under similar hypotheses to that of [Rub87, Theorem 8.4].

In order to carry out the step (2), as we do not have the non-vanishing results of [Roh84] for higher weight CM forms, we adapt instead the method of Arnold [Arn07] to prove the non-triviality of the twisted elliptic unit Euler system along the anticyclotomic tower, which in turn is based on a non-vanishing theorem of Greenberg [Gre85]. We then relate the twisted elliptic unit Euler system to Kato's Euler system of Beilinson elements (c.f., [Kat04, 15.16.1]) and make use of Kato's explicit reciprocity laws to calculate the image of Kato-Beilinson elements under the dual exponential map in terms of the relevant L-values.

Once we overcome these difficulties to settle the steps (1) and (2) above, the main results of the paper follow as a standard application of the Euler system machinery. Before we state results more precisely, we introduce some notation.

1.1. **Notation and Hypotheses.** Let $f = \sum a_n q^n \in S_w(N, \epsilon)$ be a normalized eigenform of even weight $w \geq 2$, level N and character ϵ and let $\bar{f}$ be the dual form. Fix once and for all a prime $p > 3$ such that $p \nmid N$ and $a_p = 0$. Let $F = \mathbb{Q}(\{a_n\})$ be the number field generated by the Fourier coefficients of f (equivalently, by those of $\bar{f}$). Throughout, we will assume that $F = \mathbb{Q}$ (it should be possible to remove this assumption with some work). We write ρ_f for the 2-dimensional $\mathbb{Q}_p$-representation of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to f by Deligne [Del71] and let V_f (resp. T_f) be the 2-dimensional $\mathbb{Q}_p$-vector space (resp., the $\mathbb{Z}_p$-lattice inside V_f) realizing ρ_f.

The eigenform f is called a CM form if there exists a imaginary quadratic field K and an algebraic Hecke character

$$\psi : \mathbb{A}_K/K^\times \longrightarrow \mathbb{C}^\times$$

such that f is the cusp form associated to ψ (c.f., [Kat04, §15.10]). *We assume until the end that f is a CM form* and that *K has class number 1*. Let τ denote the involution on $G_K = \mathrm{Gal}(\overline{K}/K)$ induced by complex conjugation. As in [Arn07], we assume further that $\overline{\psi} \circ \tau = \psi$; so that the sign $W(\psi) \in \{\pm 1\}$ of the functional equation of ψ (namely, the sign of the functional equation for the Hecke L-series attached to f) makes sense.

For a prime $p \nmid N$, the condition that $a_p = 0$ is equivalent to asking that p is inert in K: As otherwise, if $p = \wp\bar{\wp}$ split in K, then we would know that $a_p = \psi(\wp) + \psi(\bar{\wp})$ is a p-adic unit. Write $\mathfrak{p}$ for the unique prime of K above p. Let $K_\mathfrak{p}$ be its completion at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ its ring of integers at $\mathfrak{p}$. Denote the conductor of ψ by $\mathfrak{f} = \mathfrak{f}_\psi$ and set $K(\mathfrak{f}p^\infty) = \cup_n K(\mathfrak{f}p^n)$, where for an integral ideal $\mathfrak{a}$ of K, $K(\mathfrak{a})$ denotes the ray class field of K of conductor $\mathfrak{a}$. By the theory of complex multiplication, the action of $G_\mathbb{Q}$ on V_f factors through $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/\mathbb{Q})$. The action of G_K on the one-dimensional $K_\mathfrak{p}$-vector space V_f is given by the $\mathfrak{p}$-adic avatar $\psi_\mathfrak{p} : G_K \to \mathcal{O}_\mathfrak{p}^\times$ of the Hecke character ψ.

Let $\boldsymbol{\mu}_{p^n}$ denote the group of p^n-th roots of unity and $\boldsymbol{\mu}_{p^\infty} = \varinjlim \boldsymbol{\mu}_{p^n}$. Following [Kat04], let $V = V_f(w/2)$ denote the central critical twist of V_f and let $T = T_f(w/2)$. Set $W = V/T$. Let $V^* = \mathrm{Hom}(V, \mathbb{Q}_p(1))$, $T^* = \mathrm{Hom}(W, \boldsymbol{\mu}_{p^\infty})$ and $W^* = \mathrm{Hom}(T, \boldsymbol{\mu}_{p^\infty})$.

Let D_∞ denote the anticyclotomic $\mathbb{Z}_p$-extension of K and let D_n denote its nth layer. Let $\Gamma = \mathrm{Gal}(D_\infty/K)$ and $\Lambda = \mathcal{O}_\mathfrak{p}[[\Gamma]]$. For a torsion Λ-module M, let $\mathrm{char}(M)$ denote its characteristic ideal.

1.2. **Statements of the results.** Only in the statement of the theorem below, let ϕ denote Euler's function. Let f be a CM-form as above and let $\mathrm{Sel}(f/D_n)$ denote the $\mathfrak{p}$-adic Selmer group attached to *central critical twist* of V_f (see §2.5 below for its precise definition) over D_n. Assume the truth of Conjectures 2.5 and 2.13 (to which we have also alluded above).

Theorem A. *There is an integer e independent of n such that*

$$\mathrm{corank}_{\mathcal{O}_\mathfrak{p}}(\mathrm{Sel}(f/D_n)) = e + \sum_{\substack{1 \le k \le n \\ (-1)^k = \varepsilon}} \phi(p^k)$$

for all $n \gg 0$.

See Theorem 4.1 below for a proof of this statement, where we in fact prove considerably more. This is a theorem of Agboola and Howard (and originally a conjecture of Greenberg [Gre83, p. 247]) when f is of weight 2; and proved unconditionally in that case. Due to some shortcomings in the current state of art in p-adic Hodge theory as well as the lack of a higher weight-analogue of a non-vanishing result due to Rohrlich, we are forced to impose the additional hypotheses in order to prove Theorem A.

Let ε denote the sign of $W(\psi)$. Let $\mathrm{Sel}_{-\varepsilon}(f/D_\infty)$ denote the Iwasawa theoretic $\pm$-Selmer group attached to the central critical twist of V_f along the anticyclotomic tower; see §2.5 for its definition. Let $X_{-\varepsilon}(f/D_\infty)$ denote its Pontrygain dual, on which Λ acts according to our convention in Definition 3.3 below. Let $\mathcal{H}^1_\varepsilon \subset \varprojlim H^1(D_{n,\mathfrak{p}}, T^*)$ denote a certain submodule which we define in §2.4. Defined in §3, let $\mathcal{C}(D_\infty) \subset \mathcal{H}^1_\varepsilon$ denote the submodule obtained from twisted elliptic units.

Theorem B. *Under the assumptions of Theorem A,*

$$\mathrm{char}(X_{-\varepsilon}(f/D_\infty)) = \mathrm{char}\left(\mathcal{H}^1_\varepsilon/\mathcal{C}(D_\infty)\right).$$

This is Theorem 4.2 in the main text. One expects that the right side of the equality in Theorem B would relate to an appropriately defined anticyclotomic p-adic L-function, via Kato's explicit reciprocity law; see [AH05, Theorem 4.3] for the relevant discussion in the case $w = 2$.

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2. Selmer groups

2.1. **Some local analysis.** Let $\mathcal{V}$ be a crystalline representation of $G_{K_\mathfrak{p}}$. For a finite extension F of $K_\mathfrak{p}$, let k_F denote its residue field and F_0 the fraction field of the Witt vectors $W(\mathrm{k}_F)$ of k_F. Let $D_F(\mathcal{V}) := (B_{\mathrm{cris}} \otimes \mathcal{V})^{G_F}$ denote the Dieudonné module of $\mathcal{V}$. We simply write $D(\mathcal{V})$ in place of the $K_\mathfrak{p}$-vector space $D_{K_\mathfrak{p}}(\mathcal{V})$. Let $\{D^i(\mathcal{V})\}$ denote the de Rham filtration on $D(\mathcal{V})$ and let

$$\exp_{n,\mathcal{V}} : D_{n,\mathfrak{p}} \otimes D(\mathcal{V})/D^0(\mathcal{V}) \longrightarrow H^1(D_{n,\mathfrak{p}}, \mathcal{V})$$

denote the $K_\mathfrak{p}[\Gamma_n]$-equivariant Bloch-Kato exponential map, defined as in [BK90]. If one assumes

(2.1) the eigenvalues of the crystalline Frobenius φ on $D(\mathcal{V})$ are not integral powers of p,

then as explained in [BK90, Theorem 4.1], the exponential map is in fact an injection

$$\exp_{n,\mathcal{V}(j)} : D_{n,\mathfrak{p}} \otimes D(\mathcal{V}(j))/D^0(\mathcal{V}(j)) \hookrightarrow H^1(D_{n,\mathfrak{p}}, \mathcal{V}(j))$$

for every integer j, and its image is denoted by $H^1_f(D_{n,\mathfrak{p}}, \mathcal{V}(j))$.

The representation V_f attached to f is crystalline and its de Rham filtration is given by

$$D^i(V_f) = \begin{cases} D(V_f) & , \quad i \le 0 \\ K_\mathfrak{p}\,\omega & , \quad 1 \le i \le w-1 \\ 0 & , \quad i \ge k \end{cases}$$

where $0 \ne \omega \in D(V_f)$. The action φ of the crystalline Frobenius satisfies $\varphi^2 + \varepsilon(p)p^{w-1} = 0$ (as $a_p = 0$). Since we assumed that the weight w is even, we conclude that V_f satisfies the hypothesis (2.1). In particular, the Bloch-Kato exponential map induces a $K_\mathfrak{p}[\Gamma_n]$-isomorphism

$$\exp_{n,V} : D_{n,\mathfrak{p}} \otimes D(V)/D^0(V) \xrightarrow{\sim} H^1_f(D_{n,\mathfrak{p}}, V)$$

for the central critical twist V. Fixing a choice of ω as above, we obtain an isomorphism

$$\exp^{(\omega)}_{n,V} : D_{n,\mathfrak{p}} \xrightarrow{\sim} H^1_f(D_{n,\mathfrak{p}}, V)$$

of $K_\mathfrak{p}[\Gamma_n]$-modules, where the corestriction map

$$\mathrm{cor}_{D_{m,\mathfrak{p}}/D_{n,\mathfrak{p}}} : H^1_f(D_{m,\mathfrak{p}}, V) \longrightarrow H^1_f(D_{n,\mathfrak{p}}, V)$$

for $m \ge n$ corresponds to the trace map

$$\mathrm{Tr}_{m/n} : D_{m,\mathfrak{p}} \longrightarrow D_{n,\mathfrak{p}}$$

on the left. We define $H^1_f(D_{n,\mathfrak{p}},W)$ (reps., $H^1_f(D_{n,\mathfrak{p}},T)$) as the direct (reps., inverse) image of $H^1_f(D_{n,\mathfrak{p}},V)$ under the natural map induced from the exact sequence

$$0 \longrightarrow T \longrightarrow V \longrightarrow W \longrightarrow 0.$$

Define $H^1_f(D_{n,\mathfrak{p}},T^*)$ (resp., $H^1_f(K,W^*)$) to be the orthogonal compliment of $H^1_f(D_{n,\mathfrak{p}},W)$ (resp., of $H^1_f(D_{n,\mathfrak{p}},T)$) under the local Tate pairing.

Lemma 2.1. $H^0(D_{\infty,\mathfrak{p}},W)=0=H^0(D_{\infty,\mathfrak{p}},W^*)$.

Proof. The proof that $H^0(K_{\mathfrak{p}},W)=0=H^0(K_{\mathfrak{p}},W^*)$ follows from the proof of [Lei11, Lemma 4.4]. As Γ_n is a non-trivial p-group for $n\geq 1$, we observe that

$$\#H^0(D_{n,\mathfrak{p}},X)\equiv \#H^0(K_{\mathfrak{p}},X) \mod p$$

for $X=W,W^*$. The proof of the lemma follows. □

2.2. **A conjecture on universal norms.** Let K_{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of K and K_n is its nth layer. It follows from [PR00, Theorem 0.6] that

$$\varinjlim H^1_f(K_{n,\mathfrak{p}},W)=\varinjlim H^1(K_{n,\mathfrak{p}},W),$$

or equivalently, that

$$\varprojlim H^1_f(K_{n,\mathfrak{p}},T^*)=0,$$

generalizing [Rub85, Theorem 2.1] (only along the cyclotomic $\mathbb{Z}_p$-tower); see also [Lei11, Lemma 7.1]. The following could be thought of as a 'relative' version of Lei's result and may be proved without much difficulty:

Lemma 2.2. *For any integer m,*

$$\varprojlim_n H^1_f(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},T^*)=0.$$

Proof. By the definition of Bloch-Kato subgroups, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1_f(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},T^*) \arrow[r] & H^1(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},T^*) \arrow[r] \arrow[d] & H^1(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},B_{\mathrm{cris}}\otimes V^*) \arrow[d] \\
0 \arrow[r] & H^1_f(K_{n,\mathfrak{p}},T^*) \arrow[r] & H^1(K_{n,\mathfrak{p}},T^*) \arrow[r] & H^1(K_{n,\mathfrak{p}},B_{\mathrm{cris}}\otimes V^*)
\end{tikzcd}$$

where the vertical arrows are corestriction maps. We therefore have an induced map

$$H^1_f(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},T^*)\longrightarrow H^1_f(K_{n,\mathfrak{p}},T^*). \tag{2.2}$$

Consider the map

$$\varprojlim_s H^1(D_{s,\mathfrak{p}}K_{n,\mathfrak{p}},T^*)\longrightarrow H^1(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},T^*) \tag{2.3}$$

whose cokernel is $H^2(K_{n,\mathfrak{p}},T^*\otimes\Lambda)[\gamma^{p^m}-1]$. It follows from the proof of [Lei11, Lemma 4.4] that $H^0(K_{n,\mathfrak{p}},W)=0$. By local duality, this means $H^2(K_{n,\mathfrak{p}},T^*)=0$, which by Nakayama's Lemma (together with the fact that the cohomological dimension of $G_{K_{n,\mathfrak{p}}}$ is 2) shows that $H^2(K_{n,\mathfrak{p}},T^*\otimes\Lambda)=0$. We therefore proved that the map (2.3) is surjective. This shows that the map (therefore also the map (2.2))

$$H^1(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}},T^*)\longrightarrow H^1(K_{n,\mathfrak{p}},T^*)$$

is induced from reduction $\mod \gamma - 1$. Thus, the map (2.2) factors through

$$\phi_n : H^1_f(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}}, T^*)/(\gamma - 1) \longrightarrow H^1_f(K_{n,\mathfrak{p}}, T^*).$$

The proof follows passing to inverse limit with respect to n and making use of Lei's result that $\varprojlim H^1_f(K_{n,\mathfrak{p}}, T^*) = 0$ together with Nakayama's Lemma. □

Remark 2.3. To give this rather simple proof of Lemma 2.2, we rely on the fact that the $G_{K_\mathfrak{p}}$-representation V^* is irreducible and it has a non-positive Hodge-Tate weight. More generally, one can compute $\varprojlim_n H^1_f(D_{m,\mathfrak{p}}K_{n,\mathfrak{p}}, T^*)$ using [Ber05] even in the absence of this strong assumption.

Note that Lemma 2.2 is essential for the definition of a Kummer pairing as in [Lei11, §7.3], based on which one may prove the following explicit reciprocity law. Let Φ be any finite extension of $\mathbb{Q}_p$. By Kummer theory, $H^1(\Phi, \mathcal{O}_\mathfrak{p}(1))$ may be identified with $\widehat{\Phi^\times} \otimes_{\mathbb{Z}_p} \mathcal{O}_\mathfrak{p}$ where $\widehat{M}$ stands for the p-adic completion of an abelian group M. Let $U_\Phi \subset \widehat{\Phi^\times} \otimes_{\mathbb{Z}_p} \mathcal{O}_\mathfrak{p}$ denote the submodule generated by the completion of units. There is a twisting isomorpshim

$$\varprojlim_{\Phi} H^1(\Phi, \mathcal{O}_\mathfrak{p}(1)) \xrightarrow{\sim} \varprojlim_{\Phi} H^1(\Phi, T^*)$$

where the inverse limit is taken over finite subextentions Φ of the unique $\mathbb{Z}_p^2$-extension of $K_\mathfrak{p}$; c.f., [Rub00, §6]. Denote the image of $\varprojlim U_\Phi$ under this isomorphism by $\mathcal{U}$. Let F be a finite subextension of $D_{\infty,\mathfrak{p}}$ and let F_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F. Let

$$\mathcal{U}_{F_{\text{cyc}}} \subset \varprojlim_{\substack{L \subset F_{\text{cyc}} \\ \text{finite over } F}} H^1(L, T^*)$$

be the image of $\mathcal{U}$ under the natural projection. Set $H^1_f(F_{\text{cyc}}, W) := \varinjlim H^1_f(L, W)$.

Lemma 2.4.

(i) *There is a Kummer pairing* $\langle\, , \rangle : H^1_f(F_{\text{cyc}}, W) \times \mathcal{U}_{F_{\text{cyc}}} \longrightarrow K_\mathfrak{p}/\mathcal{O}_\mathfrak{p}$.

(ii) *For any character χ of $\Delta = \text{Gal}(F/K_\mathfrak{p})$ and $x \in H^1_f(F, T)$, any non-negative integer m and $y = \{y_L\} \in \mathcal{U}_{F_{\text{cyc}}}$, we have*

$$\sum_{\delta \in \Delta} \chi(\delta) \langle x \otimes p^{-m}, y_F \rangle = p^{-m} \left[\sum_{\delta \in \Delta} \chi(\delta) \exp^{-1}_{F,V}(x^\delta)\, ,\, \sum_{\sigma \in \Delta} \chi^{-1}(\sigma) \exp^*_{F,V^*}(y_F^\delta) \right],$$

where $[\, ,]$ is the natural pairing on $(F \otimes D(V)) \times (F \otimes D(V^*))$ and $\exp^{-1}_{F,V}$ is the inverse of the Bloch-Kato exponential

$$\exp_{F,V} : F \otimes D(V)/D^0(V) \xrightarrow{\sim} H^1_f(F, V).$$

Proof. See the proof of [Lei11, Prop. 7.8]. □

Conjecture 2.5. $\varinjlim H^1_f(D_{n,\mathfrak{p}}, W) = \varinjlim H^1(D_{n,\mathfrak{p}}, W)$. *Equivalently,* $\varprojlim H^1_f(D_{n,\mathfrak{p}}, T^*) = 0$.

When the weight of f is 2, this is a theorem of Rubin [Rub85]. The proof of this conjecture would follow from the following extension of a conjecture of Nekovář (proved by Perrin-Riou [PR00] in the crystalline case and by Berger [Ber05] in general) to the anticyclotomic $\mathbb{Z}_p$-extension (which was originally formulated for the cyclotomic $\mathbb{Z}_p$-tower):

Conjecture 2.6. *Let $\mathcal{V}$ be a crystalline representation of $G_{K_\mathfrak{p}}$ and $\mathcal{T}$ is a $G_{K_\mathfrak{p}}$-stable lattice. Assume that (2.1) holds, as well as that*

$$H^0(K_\mathfrak{p}, \mathcal{V}/\mathcal{T}) = 0 \tag{2.4}$$

Then there is an isomorphism

$$\varprojlim H^1(D_{n,\mathfrak{p}}, \mathrm{Fil}^1\mathcal{T}) \xrightarrow{\sim} \varprojlim H^1_f(D_{n,\mathfrak{p}}, \mathcal{T}),$$

where $\mathrm{Fil}^1\mathcal{V}$ *denotes the largest sub-representation of $\mathcal{V}$ with strictly positive Hodge-Tate weights and* $\mathrm{Fil}^1\mathcal{T} = \mathcal{T} \cap \mathrm{Fil}^1\mathcal{V}$*. In particular, if $\mathcal{V}$ is irreducible with at least one non-positive Hodge-Tate weight then $\varprojlim H^1_f(D_{n,\mathfrak{p}}, \mathcal{T}) = 0$.*

As a consequence of the proof of Nekovář's conjecture alluded to above, Perrin-Riou [PR00, Theorem 0.7] deduces the following twisting result (with the notation of Conjecture 2.6):

$$\varprojlim H^1_f(K(\boldsymbol{\mu}_{p^n})_\mathfrak{p}, \mathcal{T}) \otimes \mathbb{Z}_p(1) \hookrightarrow \varprojlim H^1_f(K(\boldsymbol{\mu}_{p^n})_\mathfrak{p}, \mathcal{T}(1))$$

As a generalization of this result to anticyclotomic setting, we propose the following (which in turn could be used to reduce Conjecture 2.5 to weight 2, which is known thanks to Rubin's work [Rub85]):

Conjecture 2.7. *For any character ϕ of Γ,*

$$\varprojlim H^1_f(D_{n,\mathfrak{p}}, T^*) \otimes \phi \xrightarrow{\sim} \varprojlim H^1_f(D_{n,\mathfrak{p}}, T^* \otimes \phi).$$

Note that for any ϕ as above, both $G_{K_\mathfrak{p}}$-representations V^* and $V^* \otimes \phi$ are irreducible and both have a non-positive Hodge-Tate weight.

2.3. **Plus-Minus subgroups.** In this section, we give the definitions of the plus-minus subgroups, slightly modifying Lei's definition. For $n \geq 0$, let Ξ_n^- (resp., Ξ_n^+) denote the set of characters of Γ_n of exact order p^k with k odd (resp., k even), together with the trivial character (resp., without the trivial character). Set

$$H^1_\pm(D_{n,\mathfrak{p}}, V) = \{x \in H^1_f(D_{n,\mathfrak{p}}, V) : \sum_{\sigma \in \Gamma_n} \chi(\sigma) x^\sigma = 0, \text{ for any } \chi \in \Xi^\mp\}.$$

Set

$$D^\pm_{n,\mathfrak{p}} = \{x \in D_{n,\mathfrak{p}} : \sum_{\sigma \in \Gamma_n} \chi(\sigma) x^\sigma = 0, \text{ for any } \chi \in \Xi^\mp\},$$

so that $H^1_\pm(D_{n,\mathfrak{p}}, V)$ is the isomorphic image of $D^\pm_{n,\mathfrak{p}}$ under $\exp^{(\omega)}_{n,V}$.

Remark 2.8. Lei defines his plus-minus subgroups by setting

$$\begin{aligned} D^{+,\mathrm{Lei}}_{n,\mathfrak{p}} &= \{x \in D_{n,\mathfrak{p}} : \mathrm{Tr}_{n/m+1}(x) \in D_{m,\mathfrak{p}}, \text{ for } 0 < m \leq n \text{ even}\} \\ &= \{x \in D_{n,\mathfrak{p}} : \sum_{\sigma \in \Gamma_n} \chi(\sigma) x^\sigma = 0, \text{ for } \textit{nontrivial } \chi \in \Xi^-\}, \end{aligned}$$

$$\begin{aligned} D^{-,\mathrm{Lei}}_{n,\mathfrak{p}} &= \{x \in D_{n,\mathfrak{p}} : \mathrm{Tr}_{n/m+1}(x) \in D_{m,\mathfrak{p}}, \text{ for } 0 < m \leq n \text{ odd}\} \\ &= \{x \in D_{n,\mathfrak{p}} : \sum_{\sigma \in \Gamma_n} \chi(\sigma) x^\sigma = 0, \text{ for } \textit{nontrivial } \chi \in \Xi^+\}. \end{aligned}$$

In particular, note that $D^{-,\mathrm{Lei}}_{n,\mathfrak{p}} = D^-_{n,\mathfrak{p}}$ and that $D^+_{n,\mathfrak{p}} \subsetneq D^{+,\mathrm{Lei}}_{n,\mathfrak{p}}$.

Lemma 2.9. $D^+_{n,\mathfrak{p}} \cap D^-_{n,\mathfrak{p}} = 0$ *and* $D^+_{n,\mathfrak{p}} + D^-_{n,\mathfrak{p}} = D_{n,\mathfrak{p}}$.

Proof. Remark 2.8 and the proof of [Lei11, Lemma 4.9]) shows that $D^+_{n,\mathfrak{p}} \cap D^-_{n,\mathfrak{p}} \subset K_\mathfrak{p}$. Since Ξ_n^- contains the trivial character, we see that $K_\mathfrak{p} \cap D^+_{n,\mathfrak{p}} = 0$. This proves the first statement.

Let $\omega_n^\pm$ be the two distinguished polynomials in Λ defined as in [AH05, §5]. Then,

$$\dim_{K_\mathfrak{p}}(D^+_{n,\mathfrak{p}} + D^-_{n,\mathfrak{p}}) = \dim_{K_\mathfrak{p}}(D^+_{n,\mathfrak{p}}) + \dim_{K_\mathfrak{p}}(D^-_{n,\mathfrak{p}}) = \deg(\omega_n^+) + \deg(\omega_n^-) = \dim_{K_\mathfrak{p}}(D_{n,\mathfrak{p}}),$$

where the second equality is explained in Remark 2.10 below. □

Remark 2.10. It is straightforward to see that $\omega_n^\mp(\gamma) D_{n,\mathfrak{p}} \subset D^\pm_{n,\mathfrak{p}}$. In this remark we explain why this inequality is in fact an equality. Set $\omega_n(\gamma) := \omega_n^+(\gamma)\omega_n^-(\gamma) = \gamma^{p^n} - 1$. Using the the fact that $\omega_n^\pm$ are both distinguished in the unique factorization domain Λ, we conclude that

$$\omega_n^\mp(\gamma) D_{n,\mathfrak{p}} = \omega_n^\mp(\gamma) K_\mathfrak{p}[\Gamma_n] = K_\mathfrak{p} \otimes \omega_n^\mp \Lambda/\omega_n = K_\mathfrak{p} \otimes \Lambda/\omega_n^\pm.$$

Thus $\dim D^\pm_{n,\mathfrak{p}} \geq \deg(\omega_n^\pm)$, which shows by the first part of Lemma 2.9 that

$$\dim D_{n,\mathfrak{p}} \geq \dim D^+_{n,\mathfrak{p}} + \dim D^-_{n,\mathfrak{p}} \geq \deg(\omega_n^+) + \deg(\omega_n^-) = p^n$$

Hence we proved

$$\omega_n^\mp D_{n,\mathfrak{p}} = D^\pm_{n,\mathfrak{p}} \cong K_\mathfrak{p} \otimes \Lambda/\omega_n^\pm.$$

Corollary 2.11. *We have,*

(i) $H^1_+(D_{n,\mathfrak{p}}, V) \cap H^1_-(D_{n,\mathfrak{p}}, V) = 0$,
(ii) $H^1_+(D_{n,\mathfrak{p}}, V) + H^1_-(D_{n,\mathfrak{p}}, V) = H^1_f(D_{n,\mathfrak{p}}, V)$.

We define the plus-minus subgroup $H^1_\pm(D_{n,\mathfrak{p}}, T)$ (resp. $H^1_\pm(D_{n,\mathfrak{p}}, W)$) as the inverse (resp., the direct) image of $H^1_\pm(D_{n,\mathfrak{p}}, V)$. We also define $H^1_\pm(D_{n,\mathfrak{p}}, T^*)$ to be the orthogonal compliment of $H^1_\pm(D_{n,\mathfrak{p}}, W)$ with respect to the local Tate pairing.

2.4. **Local ranks of universal norms.** We define $\mathcal{H}^1 = \varprojlim H^1(D_{n,\mathfrak{p}}, T^*)$ and let $\mathcal{H}^1_\pm = \varprojlim H^1_\pm(D_{n,\mathfrak{p}}, T^*)$. In this section, we first prove the following proposition which is analogous to [Rub87, Prop. 8.1]:

Proposition 2.12. *$\mathcal{H}^1$ is a free Λ-module of rank 2. Assuming Conjecture 2.5, both $\mathcal{H}^1_\pm$ are free of rank 1 and $\mathcal{H}^1_+ \cap \mathcal{H}^1_- = \{0\}$.*

Proof. The proof that $\mathcal{H}^1$ is a free Λ-module follows from Lemma 2.1 and the general principles established in [Nek06, Prop. 4.2.9]; c.f., [Büy10, Remark 2.8]. It also follows from Lemma 2.1 that the natural projection $\mathcal{H}^1 \to H^1(K_\mathfrak{p}, T^*)$ is surjective. One further verifies (as in [Büy09, Theorem A.8]) that the $\mathcal{O}_\mathfrak{p}$-module $H^1(K_\mathfrak{p}, T^*)$ is free of rank 2 and thus the first part of the proposition follows.

As $H^1_\pm(D_{n,\mathfrak{p}}, T^*)$ is the orthogonal compliment of $H^1_\pm(D_{n,\mathfrak{p}}, W)$ which is contained in the module $H^1_f(D_{n,\mathfrak{p}}, W)$, we see at once that $H^1_f(D_{n,\mathfrak{p}}, T^*)$ annihilates $H^1_\pm(D_{n,\mathfrak{p}}, W)$ and therefore it is contained in $H^1_\pm(D_{n,\mathfrak{p}}, T^*)$. Consider the exact sequence

$$0 \longrightarrow H^1_f(D_{n,\mathfrak{p}}, T^*) \longrightarrow H^1_\pm(D_{n,\mathfrak{p}}, T^*) \longrightarrow Q_n \longrightarrow 0$$

where $Q_n = \dfrac{H^1_\pm(D_{n,\mathfrak{p}}, T^*)}{H^1_f(D_{n,\mathfrak{p}}, T^*)}$. Passing to limit in the sequence above (using [Rub00, Prop. B.1.1]), we conclude that

$$\mathcal{H}^1_\pm \xrightarrow{\sim} \varprojlim_n Q_n \cong \varprojlim_n \mathrm{Hom}\left(\frac{H^1_f(D_{n,\mathfrak{p}}, W)}{H^1_\pm(D_{n,\mathfrak{p}}, W)}, \mathbb{Q}_p/\mathbb{Z}_p\right) \cong \mathrm{Hom}\left(\varinjlim \frac{H^1_f(D_{n,\mathfrak{p}}, W)}{H^1_\pm(D_{n,\mathfrak{p}}, W)}, \mathbb{Q}_p/\mathbb{Z}_p\right),$$

since we assumed Conjecture 2.5 and where the second isomorphism follows from local duality. Furthermore,

$$\frac{\varinjlim H^1_f(D_{n,\mathfrak{p}},W)}{\varinjlim H^1_\pm(D_{n,\mathfrak{p}},W)} \cong \frac{\varinjlim H^1(D_{n,\mathfrak{p}},W)}{\varinjlim H^1_\pm(D_{n,\mathfrak{p}},W)} \cong \frac{\varinjlim \left(H^1_+(D_{n,\mathfrak{p}},W)+H^1_-(D_{n,\mathfrak{p}},W)\right)}{\varinjlim H^1_\pm(D_{n,\mathfrak{p}},W)}$$
$$\cong \varinjlim H^1_\mp(D_{n,\mathfrak{p}},W)$$

where the first isomorphism follows from Conjecture 2.5 and the second and third from Lemma 2.9. The proof that $\mathcal{H}^1_\pm$ is free of rank one now follows from Remark 2.10.

To finish with the proof of the proposition, observe by local duality that

$$H^1_+(D_{n,\mathfrak{p}},T^*)\cap H^1_-(D_{n,\mathfrak{p}},T^*) \cong \mathrm{Hom}\left(\frac{H^1(D_{n,\mathfrak{p}},W)}{H^1_+(D_{n,\mathfrak{p}},W)+H^1_-(D_{n,\mathfrak{p}},W)},\mathbb{Q}_p/\mathbb{Z}_p\right)$$
$$\cong \mathrm{Hom}\left(\frac{H^1(D_{n,\mathfrak{p}},W)}{H^1_f(D_{n,\mathfrak{p}},W)},\mathbb{Q}_p/\mathbb{Z}_p\right).$$

Passing to limit, the final assertion of the proposition now follows from Conjecture 2.5. □

As in [Rub87, Conjecture 2.2], we conjecture that the following to holds:

Conjecture 2.13. $\mathcal{H}^1 \cong \mathcal{H}^1_+ \oplus \mathcal{H}^1_-$.

It would be interesting to know if Conjecture 2.13 follows from the known cases (c.f., [Rub87, Theorem 8.4]) in weight 2 via a variant of Conjecture 2.7.

2.5. **Definitions of the Selmer groups.** For every finite extension F of K and for every $v \nmid p$ of F, let F_v^{ur} be the maximal unramified extension of F_v and define

$$H^1_f(F_v,T^*) = \ker\left(H^1(F_v,T^*) \longrightarrow H^1(F_v^{\mathrm{ur}},V^*)\right),$$

and let $H^1_f(F_v,W)$ be the orthogonal compliment of $H^1_f(F_v,T^*)$ with respect to the local Tate pairing. We then define the following Selmer groups for $X = W, T^*$:

- The relaxed Selmer group
$$\mathrm{Sel}_{\mathrm{rel}}(F,X) = \ker\left(H^1(F,X) \longrightarrow \prod_{v\nmid p}\frac{H^1(F_v,X)}{H^1_f(F_v,X)}\right).$$
- The true Selmer group
$$\mathrm{Sel}(F,X) = \ker\left(H^1(F,X) \longrightarrow \prod\frac{H^1(F_v,X)}{H^1_f(F_v,X)}\right).$$
- The strict Selmer group
$$\mathrm{Sel}_{\mathrm{str}}(F,X) = \ker\left(H^1(F,X) \longrightarrow \prod_{v\nmid p}\frac{H^1(F_v,X)}{H^1_f(F_v,X)} \times \prod_{v\mid p} H^1(F_v,W)\right).$$
- For fields F where the plus-minus subgroups are defined, the $\pm$-Selmer group
$$\mathrm{Sel}_\pm(F,X) = \ker\left(H^1(F,X) \longrightarrow \prod_{v\nmid p}\frac{H^1(F_v,X)}{H^1_f(F_v,X)} \times \prod_{v\mid p} H^1_\pm(F_v,X)\right).$$

Observe the obvious inclusions

$$\mathrm{Sel}_{\mathrm{str}}(F,W)\subset \mathrm{Sel}_{\pm}(F,W)\subset \mathrm{Sel}(F,W)\subset \mathrm{Sel}_{\mathrm{rel}}(F,W)$$

and similarly with W replaced by T^*. If F/K is an infinite extension, we define

$$\mathrm{Sel}_*(F,W)=\varinjlim \mathrm{Sel}_*(F',W) \qquad \check{S}_*(F,T^*)=\varprojlim \mathrm{Sel}_*(F',T^*),$$

where the limits are taken with respect to the restriction and corestriction maps, over all finite subfields F' of F/K.

We remark that the Selmer group denoted by $\mathrm{Sel}(f/D_n)$ in the statement of Theorem A (resp., $\mathrm{Sel}_{\pm}(f/D_\infty)$ in Theorem B) in the Introduction is $\mathrm{Sel}(D_n,W)$ (resp., $\mathrm{Sel}_{\pm}(D_\infty,W)$) above.

3. Elliptic units and a non-trivial anticyclotomic Euler system

Let $\mathfrak{a}$ be an integral ideal of $\mathcal{O}_K$ coprime to $6p\mathfrak{f}$, and write $\mathcal{K}_\mathfrak{a}$ for the union of all ray class fields of K of conductor prime to $\mathfrak{a}$. Let $c_{\mathrm{ell},\mathfrak{a}}$ denote the Euler system of elliptic units for $(Z_p(1),\mathfrak{f}p,\mathcal{K}_\mathfrak{a})$ as in [Rub00]. Let

$$\psi_\mathfrak{p}: G_K \longrightarrow \mathrm{Aut}_{\mathcal{O}_K}(V_f/T_f)\cong \mathcal{O}_\mathfrak{p}^\times$$

be the p-adic avatar of the Hecke character associated to f by the theory of CM and let

$$\psi_\mathfrak{p}^{(w/2)}=\psi_\mathfrak{p}\otimes\chi_{\mathrm{cyc}}^{\omega/2}: G_K\longrightarrow \mathrm{Aut}_{\mathcal{O}_K}(W)$$

be its *central critical twist* following [Kat04]. As in [Rub00, §6], one may twist the Euler system $c_{\mathrm{ell},\mathfrak{a}}$ by the character $\psi_\mathfrak{p}^{-1}\otimes\chi_{\mathrm{cyc}}^{-\omega/2}$ to obtain an Euler system for $(T^*,\mathfrak{f}p,\mathcal{K}_\mathfrak{a})$. Then $c_\mathfrak{a}(F)\in \mathrm{Sel}_{\mathrm{rel}}(F,T^*)$ for every finite extension $F\subset\mathcal{K}_\mathfrak{a}$ of K. Let

$$c_\mathfrak{a}(L)=\{c_\mathfrak{a}(F')\}\in \check{S}_{\mathrm{rel}}(L,T^*)$$

for any extension $L\subset K_\infty$, where the inverse limit is over all subfields F' of L that are finite over K. Let $\mathcal{C}_\mathfrak{a}(F)$ denote the $\mathcal{O}_\mathfrak{p}[[\mathrm{Gal}(F/K)]]$-submodule of $\mathrm{Sel}_{\mathrm{rel}}(F,T^*)$ generated by $c_\mathfrak{a}(F)$ and $\mathcal{C}(F)$ the submodule generated by $\mathcal{C}_\mathfrak{a}(F)$ as $\mathfrak{a}$ varies over all ideals of $\mathcal{O}_K$ coprime to $6p\mathfrak{f}$. Recall that $\psi=\overline{\psi}\circ\tau$ (where $\tau\in\mathrm{Gal}(K/\mathbb{Q})$ is the nontrivial automorphism) so that the sign of the functional equation for the Hecke L-function of ψ makes sense and equals $W(\psi)=\pm1$. Note that this is the sign of the functional equation for $L(f,s)$ as well.

The following is the analogue of [AH05, Prop. 3.1] in our setting. Note however that since the higher-weight versions of the non-vanishing results of Rohrlich [Roh84] are not available, we adapt Arnold's approach (see [Arn07]) in order to verify the non-vanishing of $\mathcal{C}(D_\infty)$:

Proposition 3.1. *The image of $\mathcal{C}(D_\infty)$ in $\mathcal{H}^1$ is non-trivial. Furthermore, assuming the truth of Conjecture 2.13, then the image of $\mathcal{C}(D_\infty)$ in $\mathcal{H}^1$ falls in $\mathcal{H}^1_\varepsilon$ if and only if ε is the sign of $W(\psi)$.*

Proof. For a character ϕ of Γ, let the composition

$$\mathrm{Tw}_\phi:\varprojlim H^1(D_n,T^*)\xrightarrow{\sim}\varprojlim H^1(D_n,T^*\otimes\phi)\longrightarrow H^1(K,T^*\otimes\phi)$$

denote the twisting map. We will choose ϕ suitably so as to verify that $\mathrm{Tw}_\phi(c_\mathfrak{a}(D_\infty))$ has non-trivial image in $H^1(K_\mathfrak{p},T^*\otimes\phi)$.

By mimicking the the proof of [Arn07, Prop. 2.3] (which in turn relies on [Gre85]), one may find an integer d so that:

- The character $(\psi_\mathfrak{p}/\bar{\psi}_\mathfrak{p})^d$ factors through Γ,

- if $W(\psi)=1$, then for $\psi^*=\psi^{2d+1}$ and $\omega^*=(2d+1)(\omega-1)+1=2d(\omega-1)+\omega$, we have $L(\psi^*,\omega^*/2)\neq 0$.
- if $W(\psi)=-1$, then for $\psi_*=\psi^{2d-1}$ and $\omega_*=(2d-1)(\omega-1)+1=2d(\omega-1)+2-\omega$, we have $L(\psi_*,\omega_*/2)\neq 0$.

Assume first that $W(\psi)=1$ and let $\Psi=\psi^*$ (resp., $\boldsymbol{\omega}=\omega^*$ and $\phi=(\psi_{\mathfrak{p}}/\bar{\psi}_{\mathfrak{p}})^d$). Note that ϕ factors through Γ by our choices and running assumptions. Define T_Ψ (resp., T_Ψ^*) to be the free $\mathcal{O}_{\mathfrak{p}}$-module of rank one on which G_K acts by $\Psi_{\mathfrak{p}}\otimes\chi_{\mathrm{cyc}}$ (resp., $\Psi_{\mathfrak{p}}^{-1}$). As in [Lei11], let $z^{\mathrm{Kato}}\in\varprojlim H^1(K(\boldsymbol{\mu}_{p^n})_{\mathfrak{p}},T_\Psi^*)$ be the element obtained from Kato's Euler system. Then,

$$\mathrm{Col}^\pm(z^{\mathrm{Kato}})=L_p^\pm,$$

where $\mathrm{Col}^\pm$ are the plus-minus Coleman maps defined in [Lei11, §3.4] and $L_p^\pm$ are the plus-minus *cyclotomic* p-adic L-functions attached to the theta series of the Hecke character Ψ, defined as in the displayed equations (4) and (5) of loc.cit. The interpolation property of the p-adic L-function (c.f., [Lei11] and [Pol03]) then shows that

$$\chi_{\mathrm{cyc}}^{\boldsymbol{\omega}/2-1}(\mathrm{Col}^+(z^{\mathrm{Kato}}))=\chi_{\mathrm{cyc}}^{\boldsymbol{\omega}/2-1}(L_p^+)=\Omega_\Psi^{-1}L(\Psi,\boldsymbol{\omega}/2)\neq 0. \tag{3.1}$$

where Ω_Ψ is a non-zero complex number whose exact value we need not know. Let $V_\Psi=T_\Psi\otimes\mathbb{Q}_p$ (similarly V_Ψ^*) be the associated two dimensional $\mathbb{Q}_p$-vector space. For $\eta_+\in D(V_\Psi^*)$ defined as in [Lei11, §3.5.1], one constructs the extended logarithm $\mathcal{L}_{\eta^+}$ (see §3.2 of loc. cit. for a precise definition) making use Perrin-Riou's map [PR94] as well as a logarithm map $\log^+_{p,\boldsymbol{\omega}}$ defined in [Pol03]. Then

$$\mathrm{Col}^+(z^{\mathrm{Kato}})=\mathcal{L}_{\eta^+}(z^{\mathrm{Kato}})/\log^+_{p,\boldsymbol{w}},$$

which in turn shows thanks to (3.1) that $\chi_{\mathrm{cyc}}^{\boldsymbol{\omega}/2-1}\left(\mathcal{L}_{\eta^+}(z^{\mathrm{Kato}})\right)\neq 0$.

To ease notation (and facilitate the comparison of our arguments to that of Lei), we set $r=\boldsymbol{\omega}/2-1$. Let z_{-r} be the image of z^{Kato} under the composition

$$\varprojlim_n H^1(K(\boldsymbol{\mu}_{p^n})_{\mathfrak{p}},T_\Psi^*)\xrightarrow{(-1)^r\,\mathrm{Tw}_{\chi_{\mathrm{cyc}}^{-r}}}H^1(K_{\mathfrak{p}},T_\Psi^*(-r))\,.$$

Let $M^{\mathfrak{D}}=\mathrm{Hom}(M,\mathbb{Q}_p)$ (resp., $M^*=M^{\mathfrak{D}}(1)$) stand for the linear dual (resp., Cartier dual) of a $\mathbb{Q}_p$-vector space M. Note then that $V_\Psi^*(-r)\cong V_\Psi(r+1)^{\mathfrak{D}}(1)\cong V_\Psi(r+1)^*$. Let

$$\exp^*_{r+1}:H^1(K_{\mathfrak{p}},V_\Psi(r+1)^*)\longrightarrow D^0(V_\Psi(r+1)^*)$$

be the dual exponential map and let $[\,,\,]$ be the natural pairing on $D(V_\Psi(r+1))\otimes D(V_\Psi(r+1)^*)$. Kurihara's calculation [Kur02] yields

$$\chi_{\mathrm{cyc}}^r\left(\mathcal{L}_{\eta^+}(z^{\mathrm{Kato}})\right)=r!\,\left[\left(1-\frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}(\eta^+_{r+1})\,,\,\exp^*_{r+1}(z_{-r})\right], \tag{3.2}$$

where η^+_{r+1} is the image of η^+ under the canonical map $D(V_\Psi)\to D(V_\Psi(r+1))$. This shows that $z_{-r}\neq 0$, as we have verified above that the expression on the left of (3.2) is non-zero.

Similar to above, we define $c_{\mathfrak{a}}^\Psi(K(\boldsymbol{\mu}_{p^\infty}))\in\check{S}_{\mathrm{rel}}(K(\boldsymbol{\mu}_{p^\infty}),T_\Psi^*)$ and observe that the image of $c_{\mathfrak{a}}^\Psi(K(\boldsymbol{\mu}_{p^\infty}))$ under the composition

$$\varprojlim H^1(K(\boldsymbol{\mu}_{p^n}),T_\Psi^*)\longrightarrow\varprojlim H^1(K(\boldsymbol{\mu}_{p^n})_{\mathfrak{p}},T_\Psi^*)\xrightarrow{(-1)^r\,\mathrm{Tw}_{\chi_{\mathrm{cyc}}^{-r}}}H^1(K_{\mathfrak{p}},T_\Psi(\boldsymbol{\omega}/2)^*) \tag{3.3}$$

agrees with the image of $c_{\mathfrak{a}}(D_\infty)$ under the composition

$$\varprojlim H^1(D_n,T^*) \longrightarrow \varprojlim H^1(D_{n,\mathfrak{p}},T^*) \xrightarrow{(-1)^r\,\mathrm{Tw}_\phi} H^1(K_{\mathfrak{p}},T_\Psi(\boldsymbol{\omega}/2)^*)\ . \tag{3.4}$$

Furthermore, [Kat04, 15.6.1] shows that the image of $c_{\mathfrak{a}}^{\Psi}(K(\boldsymbol{\mu}_{p^\infty}))$ under (3.3) agrees with z_{-r} (up to a non-zero factor), in particular the image of $c_{\mathfrak{a}}(D_\infty)$ inside $\mathcal{H}^1$ is non-zero. This completes the proof of the first part of the proposition when $W(\psi)=1$. In case $W(\psi)=-1$, we replace ϕ by $\phi\circ\tau$ and reduce similarly to the non-vanishing of $L(\psi_*,w_*/2)$.

To prove the second part, it suffices to prove that the image of $\mathcal{C}(D_\infty)$ inside $\mathcal{H}^1_{-\varepsilon}$ is trivial (as we assumed the truth of Conjecture 2.13). This however follows as in the proof of [AH05, Prop. 3.1], using Kato's generalized explicit reciprocity law (see [Kat99, Kat04] and [Lei11, Prop. 7.8], where the latter may be generalized using Lemma 2.4 to cover characters of Γ) and since $L(\chi\psi,\omega/2)=0$ for all $\chi\in\Xi^\varepsilon$ thanks to Greenberg's formula [Gre83, p. 247] for the sign of the functional equation, which is proved for weight 2 modular forms in loc. cit. but easily generalizes to our case of interest using Weil's formula for root numbers, stated as in [Arn07, Prop. 2.4]. ☐

Remark 3.2. The proof of Proposition 3.1 is the only place where we need Conjecture 2.13 in an essential way in order to prove our main result (that is, Theorem 4.1). If the non-vanishing results of Rohrlich were available in our setting, we could then follow the proof of [AH05, Proposition 3.1], without any need of Conjecture 2.13.

Definition 3.3. Let $X_*(D_\infty,W)=\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_*(D_\infty),\mathbb{Q}_p/\mathbb{Z}_p)$ for $*=\{\mathrm{rel},\mathrm{str},\pm,\varnothing\}$. We adopt the convention of [AH05] that Λ acts on $X_*(D_\infty,W)$ by the rule $\lambda\cdot f(x)=f(\lambda^\iota x)$, where $\iota:\Lambda\to\Lambda$ is the involution on Λ induced by inversion on group-like elements.

Proposition 3.4.

(i) *The Λ-module $\check{S}_{\mathrm{rel}}(D_\infty,T^*)$ (resp., $X_{\mathrm{str}}(D_\infty,W)$) is torsion-free of rank one (resp., torsion).*

(ii) *The Λ-module $X_{\mathrm{rel}}(D_\infty,W)$ is of rank one.*

(iii) *We have*

$$\mathrm{char}\,(X_{\mathrm{str}}(D_\infty,W))=\mathrm{char}(\check{S}_{\mathrm{rel}}(D_\infty,T^*)/\mathcal{C}(D_\infty)).$$

Proof. (i) may be proved using the Euler system machinery as in [Rub00, §II.3] and the non-triviality of $\mathcal{C}(D_\infty)$. (ii) follows from Poitou-Tate global duality as in [PR92, Prop. 4.2.3] and Lemma 2.1. (iii) follows from the 2-variable main conjecture for K (c.f., [Rub91, Theorem 4.1(i)]) exactly as in [Arn07, §3.3]. ☐

Theorem 3.5.

(i) $\mathrm{rank}_\Lambda(\check{S}_\pm(D_\infty,T^*))=\mathrm{rank}_\Lambda(X_\pm(D_\infty,W))$.

(ii) *Assuming that Conjecture 2.13 holds true, $X_\varepsilon(D_\infty,W)$ has Λ-rank one and $X_{-\varepsilon}(D_\infty,W)$ is Λ-torsion, where ε is the sign of $W(\psi)$.*

Remark 3.6. Note in particular that, it follows from Theorem 3.5 and the fact that $\check{S}_{\mathrm{rel}}(D_\infty,T^*)$ is torsion-free (Proposition 3.4) that $\check{S}_{\mathrm{str}}(D_\infty,T^*)=\check{S}_{-\varepsilon}(D_\infty,T^*)=0$, assuming Conjecture 2.13.

Proof of Theorem 3.5. Poitou-Tate global duality yields an exact sequence

$$0 \to \check{S}_{\pm}(D_\infty, T^*) \to \check{S}_{\mathrm{rel}}(D_\infty, T^*) \to \mathcal{H}^1/\mathcal{H}^1_{\pm} \to X_{\pm}(D_\infty, W) \to X_{\mathrm{str}}(D_\infty, W) \to 0, \tag{3.5}$$

which proves (i) using Propositions 2.12 and 3.4. Assuming the truth of Conjecture 2.13, we note that $\mathcal{C}(D_\infty) \subset \check{S}_\varepsilon(D_\infty, T^*)$ by Proposition 3.1. The rest may be proved following the proof of [AH05, Theorem 3.6] *verbatim*.

□

4. Applications

4.1. **The variation of Selmer ranks.** In [AH05, §5], the authors obtain the anticyclotomic analogues of Kobayashi's control theorems [Kob03, Theorem 9.3]. These extend without further difficulty to apply in our setting as well and may be used along with Theorem 3.5 to prove the following result on the variation of the ranks of Selmer ranks along the anticyclotomic tower.

Let $Y_\pm$ denote the Λ-torsion submodule of $X_\pm(D_\infty, W)$. We assume the truth of Conjectures 2.5 and 2.13.

Theorem 4.1. *Let ε denote the sign of $W(\psi)$. Then*

$$\mathrm{corank}_{\mathcal{O}_\mathfrak{p}}\left(\mathrm{Sel}(D_n, W)\right) = \mathrm{rank}_{\mathcal{O}_\mathfrak{p}}\left(\Lambda/\omega_n^\varepsilon\right) + \mathrm{rank}_{\mathcal{O}_\mathfrak{p}}(Y_+/\omega_n^+) + \mathrm{rank}_{\mathcal{O}_\mathfrak{p}}(Y_-/\omega_n^-)$$

for all n.

4.2. **An anticyclotomic main conjecture.** The theorem we state next may be thought of as a form of an anticyclotomic main conjecture (without the p-adic L-function). The assumptions under which Theorem 4.1 holds are in effect here as well.

Theorem 4.2. *Let ε be the sign of $W(\psi)$. We then have an equality of characteristic ideals*

$$\mathrm{char}(X_{-\varepsilon}(D_\infty, W)) = \mathrm{char}\left(\mathcal{H}^1_\varepsilon/\mathcal{C}(D_\infty)\right).$$

Proof. The exact sequence (3.5) yields an injection

$$\check{S}_{\mathrm{rel}}(D_\infty, T^*)/\check{S}_\varepsilon(D_\infty, T^*) \hookrightarrow \mathcal{H}^1/\mathcal{H}^1_\varepsilon \cong \mathcal{H}^1_{-\varepsilon}.$$

Since both $\check{S}_{\mathrm{rel}}(D_\infty, T^*)$ and $\check{S}_\varepsilon(D_\infty, T^*)$ are Λ-modules of rank one (Theorem 3.5) and $\mathcal{H}^1_{-\varepsilon}$ is Λ-torsion free, it follows that $\check{S}_{\mathrm{rel}}(D_\infty, T^*) = \check{S}_\varepsilon(D_\infty, T^*)$. Furthermore, as $\check{S}_{-\varepsilon}(D_\infty, T^*) = 0$ (Remark 3.6), the exact sequence (3.5) reduces to the exact sequence

$$0 \longrightarrow \mathcal{H}^1_\varepsilon/\check{S}_\varepsilon(D_\infty, T^*) \longrightarrow X_{-\varepsilon}(D_\infty, W) \longrightarrow X_{\mathrm{str}}(D_\infty, W) \longrightarrow 0.$$

This combined with the exact sequence

$$0 \longrightarrow \check{S}_\varepsilon(D_\infty, T^*)/\mathcal{C}(D_\infty) \longrightarrow \mathcal{H}^1_\varepsilon/\mathcal{C}(D_\infty) \longrightarrow \mathcal{H}^1_\varepsilon/\check{S}_\varepsilon(D_\infty, T^*) \longrightarrow 0$$

along with Proposition 3.4(iii) proves the theorem. □

One expects that the right side of the equality in Theorem 4.2 would relate to an appropriately defined anticyclotomic p-adic L-function, via Kato's explicit reciprocity law; see [AH05, Theorem 4.3] for the relevant discussion in the case $w = 2$.

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